CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

Number of problems—6

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

Notes: (1) In this examination, $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. Let f be the function given by $f(x) = x^3 - 7x + 6$.

   (a) Find the zeros of f.

   (b) Write an equation of the line tangent to the graph of f at $x = -1$.

   (c) Find the number c that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1, 3]$.

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2. Let R be the region in the first quadrant enclosed by the graph of $y = \sqrt{6x + 4}$, the line $y = 2x$, and the y-axis.

   (a) Find the area of R.

   (b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the x-axis.

   (c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the y-axis.

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3. A particle moves along the x-axis in such a way that its acceleration at time t for $t \geq 0$ is given by $a(t) = 4 \cos(2t)$. At time $t = 0$, the velocity of the particle is $v(0) = 1$ and its position is $x(0) = 0$.

   (a) Write an equation for the velocity $v(t)$ of the particle.

   (b) Write an equation for the position $x(t)$ of the particle.

   (c) For what values of t, $0 \leq t \leq \pi$, is the particle at rest?

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4. Let f be the function given by $f(x) = \dfrac{x}{\sqrt{x^2 - 4}}$.

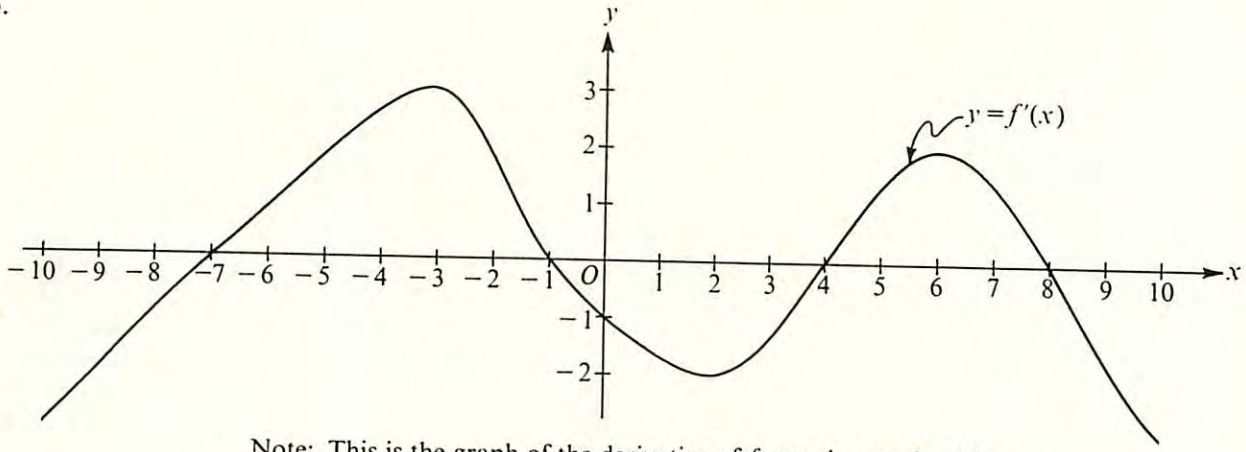
(a) Find the domain of f.

(b) Write an equation for each vertical asymptote to the graph of f.

(c) Write an equation for each horizontal asymptote to the graph of f.

(d) Find $f'(x)$.

5.



Note: This is the graph of the derivative of f, not the graph of f.

The figure above shows the graph of f', the derivative of a function f. The domain of f is the set of all real numbers x such that $-10 \leqq x \leqq 10$.

(a) For what values of x does the graph of f have a horizontal tangent?

(b) For what values of x in the interval $(-10, 10)$ does f have a relative maximum? Justify your answer.

(c) For what values of x is the graph of f concave downward?

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6. Oil is being pumped continuously from a certain oil well at a rate proportional to the amount of oil left in the well; that is, $\dfrac{dy}{dt} = ky$, where y is the amount of oil left in the well at any time t. Initially there were 1,000,000 gallons of oil in the well, and 6 years later there were 500,000 gallons remaining. It will no longer be profitable to pump oil when there are fewer than 50,000 gallons remaining.

(a) Write an equation for y, the amount of oil remaining in the well at any time t.

(b) At what rate is the amount of oil in the well decreasing when there are 600,000 gallons of oil remaining?

(c) In order not to lose money, at what time t should oil no longer be pumped from the well?

END OF EXAMINATION.

1989

1989
(1)

$f(x) = x^3 - 7x + 6$

(3) A)

	1	0	−7	6
1 ⌋	1	1	−6	0

$x^2 + x - 6$

$(x-2)(x+3)$ (2)

Zeros of $f(x)$: $x \in \{-3, 1, 2\}$

(3) B)

	1	0	−7	6
−1 ⌋	1	−1	−6	12 = f(−1)

$-1 \rfloor \; 1 \;\; -2 \;\; -4 = m$ (2)

$y - y_1 = m(x - x_1)$

$y - 12 = -4(x+1)$

$y - 12 = -4x - 4$

$y = -4x + 8$ (1)

(9) C) $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ (1)

w↑f c

$f'(x) = \dfrac{12 - 0}{3 - 1} = 6$ (1)

	1	0	−7	6
1 ⌋	1	0	−6	0
3 ⌋	1	3	2	12

(2)

$f'(x) = 3x^2 - 7 \overset{\text{set}}{=} 6$ (1)

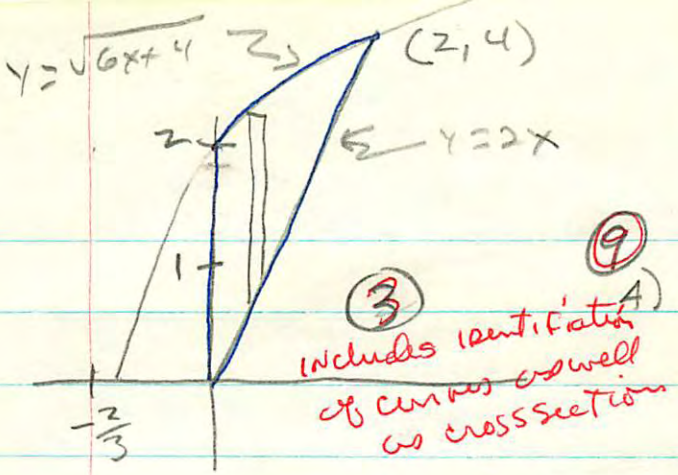
$3x^2 = 13$

$x = \pm\sqrt{\frac{13}{3}} = \frac{\pm\sqrt{39}}{3}$ (1)

$\frac{13}{3}$ is just over 4

so $\sqrt{\frac{13}{3}}$ is just larger than 2 so it is (3)

Reject neg answer, not in interval [1, 3] within the interval



(9)

(3) includes identification of curves as well as cross section

4)

$2x \overset{\text{let}}{=} \sqrt{6x+4}$ (1)

$4x^2 = 6x+4$

$2x^2-3x-2=0$

$(2x+1)(x-2)$

$x=-\frac{1}{2}$ $x=2$ (2,4)

Reject not in 1st Quad (1) Pt of intersection (1)

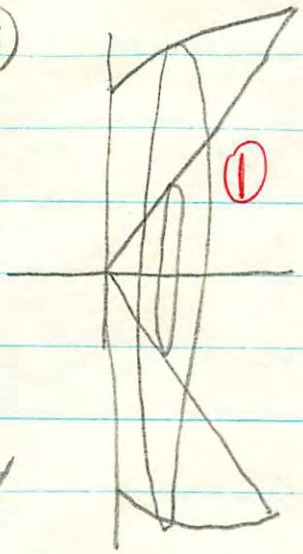
(1) $\int_0^2 \sqrt{6x+4} - 2x\,dx$

$\frac{1}{6}\frac{2}{3}(6x+4)^{\frac{3}{2}} - x^2 \Big|_0^2$ (1)

$\frac{1}{9}(64) - 4 - \frac{1}{9}(8) + 0$

$\frac{64-36-8}{9}$ $\frac{20}{9}$ sq units Area of R (1)

(3) B)



Washers. (1) $\pi\int R_2^2 - R_1^2\,dx$

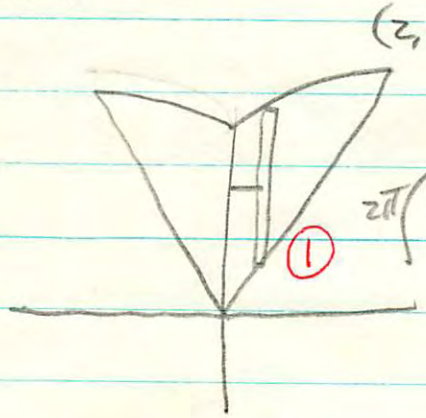
$\pi\int_0^2 6x+4-4x^2\,dx$ (1)

$y^2 = 6x+4$

$y^2-4 = 6x$

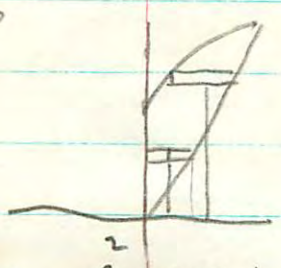
$\frac{1}{6}(y^2-4) = x$

3 C)



using Shells. (1)

2π(Radius length. dx



$\int_0^2 y\left(\frac{y}{2}\right)dy +$

$\int_2^4 y\left(\frac{y}{2} - \frac{1}{6}(y^2-4)\right)dy$

$2\pi\int_0^2 x\left(\sqrt{6x+4} - 2x\right)dx$ (1)

↑ Radius of Rev. (1)

(5)

(5) A) $a(t) = 4\cos 2t$

$v(t) = \int a(t)\,dt$ (1)

$= \int 4\cos 2t\,dt$

$v(t) = \frac{1}{2}4\sin 2t + c$ (1) (1)

$v(t) = 2\sin 2t + c$

$v(1) = 2\sin 0 + c \overset{\text{set}}{=} 1$ (1)

$\therefore c = 1$

$v(t) = 2\sin 2t + 1$ (1)

B)

$x(t) = \int v(t)\,dt$ (1)

$= \int 2\sin 2t + 1\,dt$

$x(t) = -\frac{1}{2}\cos 2t + t + c$ (1) (1)

$x(0) = -1 + 0 + c \overset{\text{set}}{=} 0$ (1)

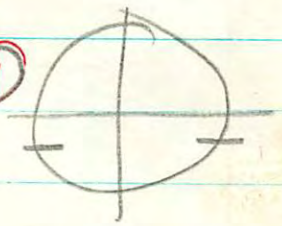
$c = 1$

$x(t) = -\cos 2t + t + 1$ (1)

(5) C) Particle at rest when $v(t) = 0$ (1)

$2\sin 2t + 1 \overset{\text{set}}{=} 0$ (1)

$\sin 2t = -\frac{1}{2}$


(1)

$0 \le t \le \pi$

$0 \le 2t \le 2\pi$

$2t \in \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

$t \in \left\{\frac{7\pi}{12}, \frac{11\pi}{12}\right\}$ (2)

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(4)

$$f(x) = \frac{x}{\sqrt{x^2-4}}$$

(4) a) D_f $\quad x^2-4>0$ (1) $\quad (x-2)(x+2)>0$

(1)

$+ \; 0 \; - \; 0 \; +$

$-2 \quad 0 \quad 2$

$D_f \quad x \in \mathbb{R}, \; x<-2$ or $x>2$ (2)

(4) B) vertical asympote are non reducible domain restrictions. (2)

$\lim_{x\to 2^+} f(x) = \frac{2}{0} \Rightarrow +\infty. \quad \lim_{x\to -2^-} f(x) \Rightarrow -\infty$

(2)

asympotes: $x=2, \; x=-2$

(4) c) $y = \frac{x}{\sqrt{x^2-4}}$

$x = \frac{z}{\sqrt{z^2-4}}$

$x^2z^2 - 4x^2 = z^2$ (2) $\rightarrow z^2(x^2-1) = 4x^2$

$z^2 = \frac{4x^2}{x^2-1}$ $\quad D_f$ $x \notin \{\pm 1\}$

$z = \sqrt{\frac{4x^2}{x^2-1}}$ (2)

$+ \; 0 \; - \; 0 \; - \; 0 \; +$

$-1 \quad 0 \quad 1$

$\therefore$ Range of original does not include ± 1

Thus $y=1, \; y=-1$ are horizontal asymptotes

(3) D) $f(x) = \frac{x}{\sqrt{x^2-4}}$ (1)

$$f'(x) = \frac{\sqrt{x^2-4} - x\left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2-4}}\right)}{x^2-4} \quad (1)$$

$$= \frac{x^2-4-x^2}{(x^2-4)^{3/2}}$$

$$= \frac{-4}{(x^2-4)^{3/2}} \quad (1)$$

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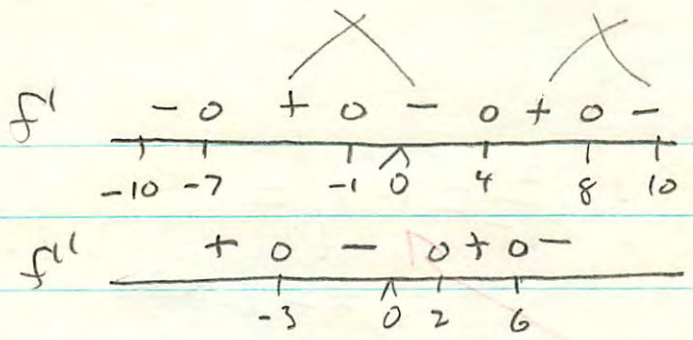
(5)



(5) A) f has horizontal tangent when $f' = 0$ (2)

$\therefore x \in \{-7, -1, 4, 8\}$ (3)

(5) B) Max occurs $x \in \{-1, 8\}$ are relmax by f' test, (3)

$(-10, f(-10))$ is also a rel max because f decreases from it. (2)

But -10 is not in the open interval $(-10, 10)$

(5) C) f is concave down when $f'' < 0$ * (1)

$x = 6$ is a critical value of f', ie $f''(6) = 0$. $x = 6$ is a max value of f' $\therefore f''(6-\varepsilon) > 0$ and $f''(6+\varepsilon) < 0$

(2) So we are now able to find in all remaining intervals (alternating signs)

$f'' < 0$: $-3 < x < 2$ or $6 < x < 10$

Thus f is concave down for those intervals.

* critical values of f' ie zeros of f'' are extremum of f' $\therefore$ critical values of f': $x \in \{-3, 2, 6\}$ (2)

Other Reasoning: f concave down when $f'' < 0$. $f'' < 0$ when f' decreasing. (1) (2)

#6 method #2

A) $\frac{dy}{dt} = ky$

$\int \frac{dy}{y} = \int k\,dt$

$\ln y = kt + C$

$y = ce^{kt}$ $\quad T=0,\ y=10^6$

$y = 10^6 e^{kt}$

$T=6$

$y = 5\times10^5$

$5\times10^5 = 10^6 e^{6k}$

$\frac{1}{2} = e^{6k}$

$-\ln 2 = 6k$

$k = -\frac{1}{6}\ln 2$

$y = 10^6 e^{-\frac{t}{6}\ln 2}$

$= 10^6 e^{\ln\left(\frac{1}{2}\right)^{\frac{t}{6}}}$

$= 10^6 \left(\frac{1}{2}\right)^{\frac{t}{6}}$

B) $\frac{dy}{dt} = ky$ $\quad =$

$k = -\frac{1}{6}\ln 2$

$y = 600{,}000$

$600{,}000\left(-\frac{1}{6}\ln 2\right)$ (100,000)

$-100{,}000\ln 2$

Decreases at $100{,}000\ln 2$

C) $y = 10^6 e^{-\frac{t}{6}\ln 2}$

Set $5\times10^4 = 10^6 e^{-\frac{t}{6}\ln 2}$

$\frac{1}{20} = e^{-\frac{t}{6}\ln 2}$

$-\ln 20 = -\frac{t}{6}\ln 2$

$\frac{6\ln 20}{\ln 2} = t$

1989
(6)

$\frac{dy}{dt} = ky$

(1) $\int \frac{dy}{y} = \int k\,dt$

$\ln|y| = kt + c$

$y = e^{kt+c}$

(1) $y = ce^{kt}$

$t=0 \quad y = 1m$

$\therefore y = 1m\,e^{kt}$

$t=6 \quad y = 500{,}000$

$\therefore 500{,}000 = 1000000\,e^{6k}$

$\left(\frac{1}{2}\right)^{\frac{1}{2}} = e^{k}$ (3) Eliminated e^k

$y = 1000000\left(\frac{1}{2}\right)^{\frac{t}{6}}$

(5) A)

(7) B) $\frac{dy}{dt}\Big|$

(1) $y = 600{,}000$

Need t when y value is 600,000 left

(1) Set $600{,}000 = 1000000\left(\frac{1}{2}\right)^{\frac{t}{6}}$

$\frac{3}{5} = \left(\frac{1}{2}\right)^{\frac{t}{6}}$

$\ln 3 - \ln 5 = \frac{t}{6}\ln\left(\frac{1}{2}\right)$

(2) $\frac{6(\ln 3 - \ln 5)}{-\ln 2} = t$

$\frac{6(\ln 5 - \ln 3)}{\ln 2} = t$

Now to get $\frac{dy}{dt}$

$y = 1000000\left(\frac{1}{2}\right)^{\frac{t}{6}} = 1000000\,e^{\frac{t}{6}\ln\left(\frac{1}{2}\right)}$

$y = 1000000\,e^{\frac{t}{6}\ln\left(\frac{1}{2}\right)}$

$\frac{dy}{dt} = 1000000\left(\frac{1}{2}\right)^{\frac{t}{6}}\left[\frac{1}{6}\ln\left(\frac{1}{2}\right)\right]$ (2)

Substitute for t (1)

$\frac{dy}{dt} = 1000000\,e^{\left(-\frac{1}{6}\ln 2\right)\left(\frac{6(\ln 5 - \ln 3)}{\ln 2}\right)\cdot\frac{1}{6}\ln\left(\frac{1}{2}\right)}$

$= 1000000\,e^{(\ln 3 - \ln 5)\cdot\frac{1}{6}\ln\left(\frac{1}{2}\right)}$

$= -\frac{500000}{3}\ln 2\left[e^{\ln\frac{3}{5}}\right]$

$= -\frac{500000}{3}\ln 2\left(\frac{3}{5}\right)$

$= -100000\ln 2$ (2)

(3) (3)

C) $y = 1000000\left(\frac{1}{2}\right)^{\frac{t}{6}}$ (1) Set $= 50{,}000$

$\left(\frac{1}{2}\right)^{\frac{t}{6}} = \frac{1}{20}$

$\frac{t}{6}\ln\left(\frac{1}{2}\right) = -\ln 20$

$\frac{t}{6} = \frac{\ln 20}{\ln 2}$

$t = \frac{6\ln 20}{\ln 2}$ (2)

about 26 years

also $t = 12 + 6\log_2 5$

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

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Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

Notes: (1) In this examination $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. Let f be the function given by $f(x) = \sqrt{x^4 - 16x^2}$.

   (a) Find the domain of f.

   (b) Describe the symmetry, if any, of the graph of f.

   (c) Find $f'(x)$.

   (d) Find the slope of the line normal to the graph of f at $x = 5$.

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2. A particle moves along the x-axis so that its velocity at any time $t \geq 0$ is given by $v(t) = 1 - \sin(2\pi t)$.

   (a) Find the acceleration $a(t)$ of the particle at any time t.

   (b) Find all values of t, $0 \leq t \leq 2$, for which the particle is at rest.

   (c) Find the position $x(t)$ of the particle at any time t if $x(0) = 0$.

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3. Let R be the region in the first quadrant enclosed by the hyperbola $x^2 - y^2 = 9$, the x-axis, and the line $x = 5$.

   (a) Find the volume of the solid generated by revolving R about the x-axis.

   (b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when R is revolved about the line $x = -1$.

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4. Let f be the function defined by $f(x) = 2xe^{-x}$ for all real numbers x.

   (a) Write an equation of the horizontal asymptote for the graph of f.

   (b) Find the x-coordinate of each critical point of f. For each such x, determine whether $f(x)$ is a relative maximum, a relative minimum, or neither.

   (c) For what values of x is the graph of f concave down?

   (d) Using the results found in parts (a), (b), and (c), sketch the graph of $y = f(x)$ in the xy-plane provided below. Note: The xy-plane is provided in the pink test booklet only.

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5. Let R be the region in the first quadrant under the graph of $y = \frac{x}{x^2 + 2}$ for $0 \leqq x \leqq \sqrt{6}$.

(a) Find the area of R.

(b) If the line $x = k$ divides R into two regions of equal area, what is the value of k?

(c) What is the average value of $y = \frac{x}{x^2 + 2}$ on the interval $0 \leqq x \leqq \sqrt{6}$?

---

6. Let f be a differentiable function, defined for all real numbers x, with the following properties.

(i) $f'(x) = ax^2 + bx$

(ii) $f'(1) = 6$ and $f''(1) = 18$

(iii) $\int_1^2 f(x)dx = 18$

Find $f(x)$. Show your work.

END OF EXAMINATION.

1988

1988 (1)

#1 $f(x) = \sqrt{x^4 - 16x^2} = |x|\sqrt{x^2-16}$

(5) a) $D_f = ?$ $\quad x^4 - 16x^2 \overset{\text{set}}{\geq} 0$

$x^2(x^2-4) \geq 0$

$x^2(x-4)(x+4) \geq 0$

+	0	−	⊘	−	0	+
	−4		0		4	

(1) Since "inside" must be ≥ 0

$|x| \geq 4$, $x = 0$ (2)

OR

$x \leq -4$, $x = 0$, $x \geq 4$

(3) b) $f(-x) = f(x)$, f is even (1)

$\therefore$ reflects across y axis.

$f(-x) = \sqrt{(-x)^4 - 16(-x)^4}$

$= \sqrt{x^4 - 16x^4}$ (2)

$= f(x)$

f is even.

(3) c) $f'(x) = \dfrac{1}{2\sqrt{x^4-16x^2}} \cdot \dfrac{4x^3 - 32x}{}$ (1) der of inside

If $x > 0$

$f' = \dfrac{2(x^2-8)}{\sqrt{x^2-16}} = \dfrac{2x^3-16x}{\sqrt{x^4-16x^2}}$

If $x < 0$

$f' = \dfrac{-2(x^2-8)}{\sqrt{x^2-16}} = \dfrac{2x(x^2-8)}{|x|\sqrt{x^2-16}}$ (1) simplified

(2) Normal: $m = -\frac{3}{34}$ understood normal Neg Recip

Neg recip slope.

(4) d) $f'(5) = \dfrac{2(5)(25-8)}{5\sqrt{25-16}}$

$= \dfrac{2(17)}{3}$

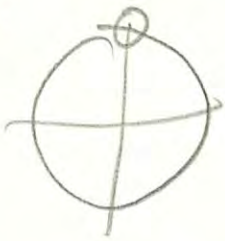
$= \dfrac{34}{3}$ (2)

$v(t) = 1 - \sin 2\pi t$

(3) A) $\frac{dv(t)}{dt} = a(t)$ (1)

$a(t) = -2\pi \cos 2\pi t$ (1) (1)

(6) B) particle at rest when $v = 0$ (1)



$v(t) = 1 - \sin 2\pi t \overset{\text{set}}{=} 0$ (1)

$\sin 2\pi t = 1$

$2\pi t = \frac{\pi}{2} + 2k\pi$

$4t = 1 + 4k$

$t = \frac{1+4k}{4} \quad k \in J$ (integers) (2)

$t \in \{\frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \ldots\}$

$0 \le t \le 2 \therefore t \in \{\frac{1}{4}, \frac{5}{4}\}$ (2)

(6) C) $x(t) = \int v(t)\,dt$ (1) $\quad x(0) = 0$

$x(t) = \int 1 - \sin 2\pi t \,dt$ (1)

$x(t) = t + \frac{1}{2\pi}\cos 2\pi t + c$

$x(0) = 0 + \frac{\cos 0}{2\pi} + c \overset{\text{set}}{=} 0$ (2)

$\frac{1}{2\pi} + c = 0$

$c = -\frac{1}{2\pi}$ (1)

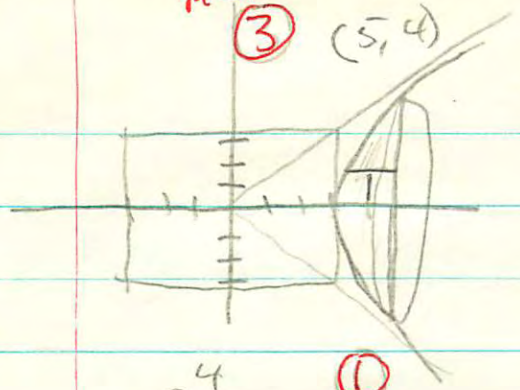
$x(t) = t + \frac{1}{2\pi}\cos 2\pi t - \frac{1}{2\pi}$ (1)

1988

(3)

(10) A)

(10)



labeled pict (3)

shells (1) radius y

(2) height $5 - x$

$x^2 - y^2 = 9$

$x = \sqrt{9 + y^2}$

$5 - \sqrt{9 + y^2}$

$5^2 - y^2 = 9$

$y^2 = 16$

(1) $y = 4$ 1st Quad $y > 0$

(1)

$$2\pi \int_0^4 y\left(5 - \sqrt{9+y^2}\right) dy =$$

Let $u = 9 + y^2$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

(1) $y = 0 \quad u = 9$

$y = 4 \quad u = 25$

$$2\pi \int_0^4 5y\,dy - 2\pi \int_0^4 y\sqrt{9+y^2}\,dy$$

$$5\pi y^2 \Big|_0^4 - 2\pi \int_9^{25} \frac{1}{2}u^{\frac{1}{2}}\,du$$

$$80\pi - \pi\left(\frac{2u^{\frac{3}{2}}}{3}\right)\Big|_9^{25} = 2\pi\left[40 - \frac{1}{3}[125 - 27]\right]$$

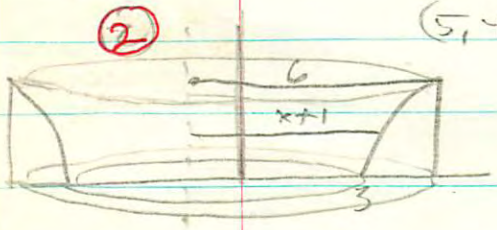
$125 - 27 = 98$

$$\frac{2\pi}{3}\left[120 - 98\right] = \frac{2\pi}{3}(22) = \frac{44\pi}{3}$$ (1)

(5)

(13) washers, $\pi \int R_1^2 - R_2^2\,dx$ (1) big Radius $= 6$ Sm Rad $= x + 1$

(2)



(1) (1)

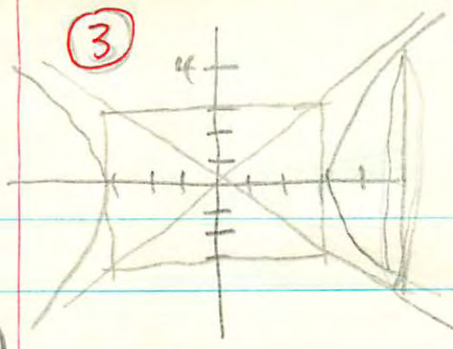
$$\pi \int_0^4 6^2 - \left(1 + \sqrt{9+y^2}\right)^2 dy$$

not required.

$$\pi \int_0^4 36 - \left[1 + 9 + y^2 + 2\sqrt{9+y^2}\right] dy$$

$$\pi \int_0^4 26 - y^2 - 2\sqrt{9+y^2}\,dy$$

1988 (3)



$x^2 - y^2 = 9$

$25 - y^2 = 9$ (1) 1st quad $\therefore y > 0$

$y = 4$

(1) using circles $\pi \int_a^b r^2 dx$

(1) typical radius $y = \sqrt{x^2 - 9}$

(10) A)

$$\pi \int_3^5 \left(\sqrt{x^2-9}\right)^2 dx = \pi \int_3^5 x^2 - 9 \, dx = \pi \left(\frac{x^3}{3} - 9x \right) \Big|_3^5 =$$

(1) (1) (1)

$$\pi \left(\frac{125}{3} - 45 \right) - \pi \left(\frac{27}{3} - 27 \right)$$

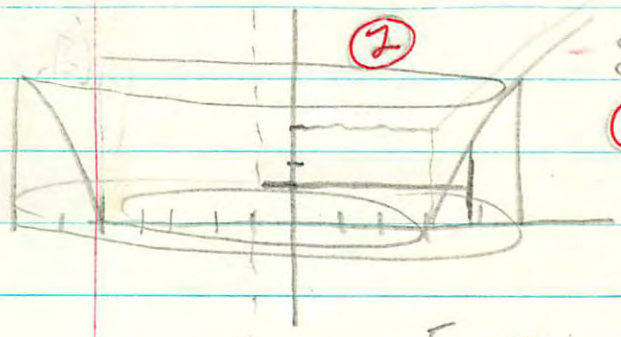
$$\pi \left(\frac{125 - 135}{3} \right) - \pi (9 - 27)$$

$$\frac{-10\pi}{3} + 18\pi$$

(1)

$$\frac{-10\pi}{3} + \frac{54\pi}{3} = \frac{44\pi}{3}$$

(5) B)



Shells: Radius $x+1$ height y

(1) $2\pi \int_a^b$ (radius) height dx

shells in dx $2\pi \int_3^5$ (1) $(x+1)$ (1) $\sqrt{x^2-9}\, dx$

$x^2 = 9 + y^2$

$2x\, dx = 2y\, dy$

$x\, dx = y\, dy$

radius $x+1$

shells in dy

$2\pi \int_0^4$ (1) $y\left(1 + \sqrt{9+y^2}\right)$ (1) $\frac{y}{\sqrt{9+y^2}}\, dy$

$1 + \sqrt{9+y^2}$ height $= y$

1988
4

$f(x) = (2x)e^{-x} = \frac{2x}{e^x}$

3

A) horizontal asymptote

$\lim_{y \to -\infty} f(x) = -\infty$ 1

$\lim_{x \to \infty} f(x) \to 0$ 1

Exponential on bottom

$y = 0$ is horizontal asymptote. 1

3

C) f'' describes concavity

$f' = \overset{u}{2e^{-x}}\overset{v}{(1-x)}$ 1

$f'' = (1-x)(-2e^{-x}) + 2e^{-x}(-1)$

$= -2e^{-x}(1-x+1)$

$= -2e^{-x}(2-x)$

Always < 0

f'': $-$ 0 $+$ 1

0 2

1

f is concave down $\forall x < 2$

5

$f(x) = \overset{u}{(2x)}\overset{v}{(e^{-x})}$ 1

B) $f'(x) = 2e^{-x} - 2xe^{-x}$

$= 2e^{-x}(1-x) \overset{\text{set}}{=} 0$

1 Always > 0

$x = 1$

f': $+$ 0 $-$ 1

0 1

1

only critical value $x = 1$

This is a max, f' test.

1

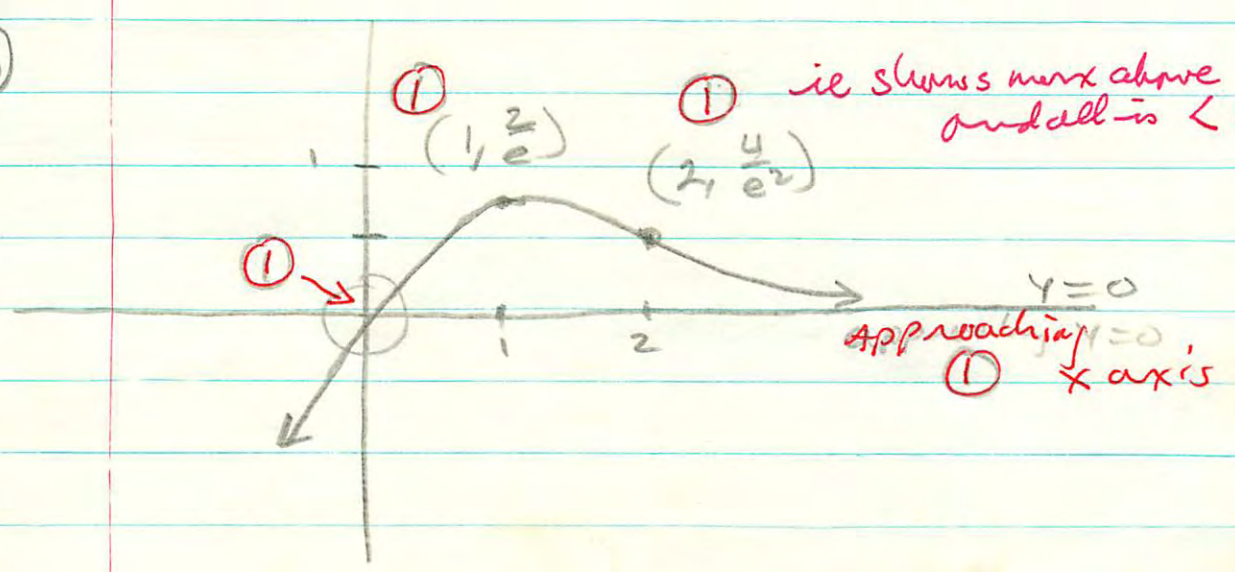
$f(1) = \frac{2}{e} \approx \frac{2}{3}$

$f(2) = \frac{4}{e^2} \approx \frac{1}{2}$

$f(0) = 0$

4

D



ie shows max above PI and all is < 1

1988 (5)

area under $y = \frac{x}{x^2+2}$ $0 \le x \le \sqrt{6}$

(6)

A) (1) $\int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx$ (1) $u = x^2+2$ $du = 2x\,dx$ $\frac{1}{2}du = x\,dx$ (1) $x=0$ $u=2$ $x=\sqrt{6}$ $u=8$

$\frac{1}{2}\int_2^8 \frac{du}{u} = \frac{1}{2}\ln u\Big|_2^8 = \frac{1}{2}[\ln 8 - \ln 2]$ (2)

$\frac{1}{2}[3\ln 2 - \ln 2] = \ln 2$ (1)

(6)

(B) Plan of attack (2)

$\int_0^K \frac{x}{x^2+2}\,dx \overset{\text{Set}}{=} \frac{1}{2}\ln 2$ let $u = x^2+2$ $\frac{1}{2}du = x\,dx$ $x=0$ $u=2$ $x=K$ $u=K^2+2$

(2) $\frac{1}{2}\int_2^{K^2+2} \frac{du}{u} = \frac{1}{2}\ln u\Big|_2^{K^2+2} = \frac{1}{2}[\ln(K^2+2) - \ln(2)]$

$\frac{1}{2}\ln(K^2+2) - \frac{1}{2}\ln(2) \overset{\text{Set}}{=} \frac{1}{2}\ln(2)$

$\frac{1}{2}\ln(K^2+2) = \ln 2$

$\ln(K^2+2) = 2\ln 2$

$= \ln 4$

$\therefore K^2+2 = 4$

$K^2 = 2$

$K = \pm\sqrt{2}$

(1) reject $-\sqrt{2}$ not in interval $\therefore K = \sqrt{2}$ (1)

(3)

(C) Average value (1) $\frac{1}{b-a}\int_a^b f(x)\,dx$

$\frac{1}{\sqrt{6}}\int_0^{\sqrt{6}} \frac{x}{x^2+2}\,dx = \frac{1}{\sqrt{6}}(\ln 2)$ From part A

(2)

1988 (6)

(15)

i) $f'(x) = ax^2 + bx$

ii) $f'(1) = 6 \quad f''(1) = 18$

iii) $\int_1^2 f(x)\,dx = 18$

find $f(x)$

$f'(x) = ax^2 + b \quad f''(x) = 2ax + b$

$f'(1) = a + b = 6 \quad f''(1) = 2a + b = 18$

$2a + b = 18$

$-a - b = -6$

$a = 12$ (2)

$12 + b = 6$ (2) $\therefore f'(x) = 12x^2 - 6x$ (1) conclusion

$b = -6$

$\int f'(x)\,dx = f(x)$

$f(x) = \int 12x^2 - 6x\,dx = 4x^3 - 3x^2 + C$ (1) (1) (1)

(3) integ this → Response in order to use given item iii

$\int 4x^3 - 3x^2 + C\,dx = x^4 - x^3 + Cx + d \Big|_1^2 = 18$ (1)

$16 - 8 + 2C + d - [1 - 1 + C + d] = 18$

$8 + 2C + d - C - d = 18$

$C = 10 \quad \therefore$ (1)

$f(x) = 4x^3 - 3x^2 + 10$ (2)

1987

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

Number of problems—6

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

Notes: (1) In this examination $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. A particle moves along the x-axis so that its acceleration at any time t is given by $a(t) = 6t - 18$. At time $t = 0$ the velocity of the particle is $v(0) = 24$, and at time $t = 1$ its position is $x(1) = 20$.

   (a) Write an expression for the velocity $v(t)$ of the particle at any time t.

   (b) For what values of t is the particle at rest?

   (c) Write an expression for the position $x(t)$ of the particle at any time t.

   (d) Find the total distance traveled by the particle from $t = 1$ to $t = 3$.

---

2. Let $f(x) = \sqrt{1 - \sin x}$.

   (a) What is the domain of f?

   (b) Find $f'(x)$.

   (c) What is the domain of f'?

   (d) Write an equation for the line tangent to the graph of f at $x = 0$.

---

3. Let R be the region enclosed by the graphs of $y = (64x)^{\frac{1}{4}}$ and $y = x$.

   (a) Find the volume of the solid generated when region R is revolved about the x-axis.

   (b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid generated when region R is revolved about the y-axis.

---

4. Let f be the function given by $f(x) = 2\ln(x^2 + 3) - x$ with domain $-3 \leq x \leq 5$.

   (a) Find the x-coordinate of each relative maximum point and each relative minimum point of f. Justify your answer.

   (b) Find the x-coordinate of each inflection point of f.

   (c) Find the absolute maximum value of $f(x)$.

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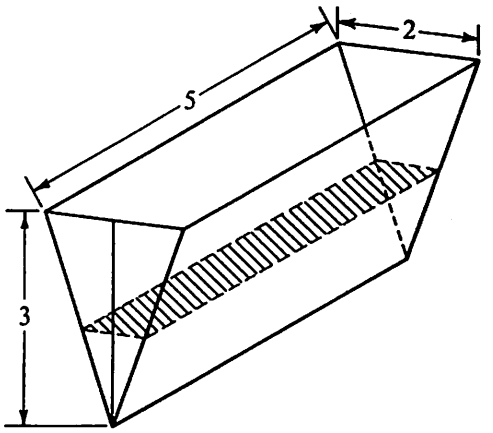
1987



5. The trough shown in the figure above is 5 feet long, and its vertical cross sections are inverted isosceles triangles with base 2 feet and height 3 feet. Water is being siphoned out of the trough at the rate of 2 cubic feet per minute. At any time t, let h be the depth and V be the volume of water in the trough.

   (a) Find the volume of water in the trough when it is full.

   (b) What is the rate of change in h at the instant when the trough is $\frac{1}{4}$ full by volume?

   (c) What is the rate of change in the area of the surface of the water (shaded in the figure) at the instant when the trough is $\frac{1}{4}$ full by volume?

---

6. Let f be a function such that $f(x) < 1$ and $f'(x) < 0$ for all x.

   (a) Suppose that $f(b) = 0$ and $a < b < c$. Write an expression involving integrals for the area of the region enclosed by the graph of f, the lines $x = a$ and $x = c$, and the x-axis.

   (b) Determine whether $g(x) = \dfrac{1}{f(x) - 1}$ is increasing or decreasing. Justify your answer.

   (c) Let h be a differentiable function such that $h'(x) < 0$ for all x. Determine whether $F(x) = h(f(x))$ is increasing or decreasing. Justify your answer.

END OF EXAMINATION

1987
#1

1

$a(t) = 6t - 18 \quad t=0, v(0)=24; \; t=1 \; x(1)=20$

(2) A) $\int a(t)\,dt = v(t) \Rightarrow \int 6t-18\,dt$ (1)

$v(t) = 3t^2 - 18t + c$

$v(0) = c \overset{\text{set}}{=} 24$ (1)

$v(t) = 3t^2 - 18t + 24$

(3) B) Particle at rest when velocity = 0

$v(t) = 3t^2 - 18t + 24 \overset{\text{set}}{=} 0$ (1)

$3(t^2 - 6t + 8) = 0$

$3(t-4)(t-2) = 0$ (1) $\quad t \in \{2, 4\}$

Particle at rest when $t=2$ (1) and when $t=4$

(2) C) $x(t) = \int v(t)\,dt \Rightarrow \int 3t^2 - 18t + 24\,dt =$

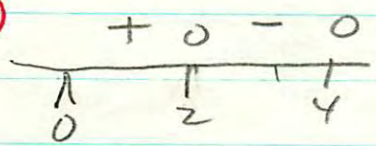
$x(t) = t^3 - 9t^2 + 24t + c$

$x(1) = 1 - 9 + 24 + c \overset{\text{set}}{=} 20$ (1)

$c = 4$

$x(t) = t^3 - 9t^2 + 24t + 4$ (1)

(3) D) v (1)



(1) Particle starts right at $t=1$. stops at 2. goes left until $t=3$

	x	-9	24	4	
4	1	−8	16	20	> 4 going right
2	1	−7	10	24	> 2 going left.
3	1	−6	6	22	

total dist traveled is 6.

(1)

1987
(2)

$f(x) = \sqrt{1-\sin x}$

(2) A) D_f? $1-\sin x \geq 0$ (1)

$\sin x \leq 1$ Range of $|\sin x| \leq 1$

$\therefore$ Domain of f, $x \in \mathbb{R}$. (1)

(3) B) $f(x) = \sqrt{1-\sin x}$

$f'(x) = \dfrac{-\cos x}{2\sqrt{1-\sin x}}$ (2) chain rule. (1)

(3) C) $D_{f'}$ $1-\sin x \neq 0$ (1)

$\sin x \neq 1$ $D_{f'}$ $x \in \mathbb{R}$, $x \notin \left\{\dfrac{(4k+1)\pi}{2}\right\}$ (2)

$\frac{\pi}{2} + 2k\pi$

$k \in J$.

(2) D) $f(0) = \sqrt{1-0} = 1$ $(0, 1)$

$f'(0) = \dfrac{-1}{2\sqrt{1}} = -\dfrac{1}{2}$ (1)

$y - 1 = -\frac{1}{2}(x-0)$

$y = -\frac{1}{2}x + 1$ (1)

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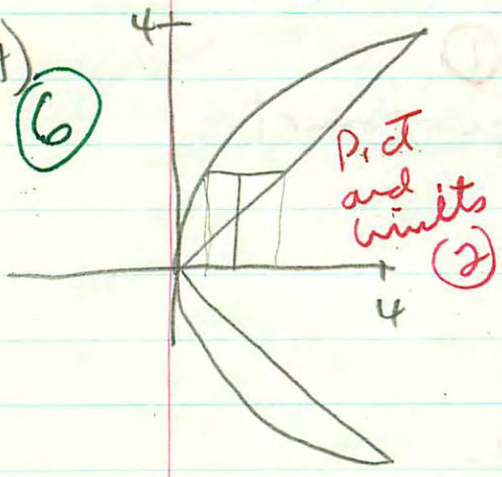
(3) using shells (1)

$y = x^{\frac{1}{4}}$

$y = (64x)^{\frac{1}{4}}$

A) (6)



P, cl and limits (2)

4 radius (1)

$$2\pi \int_0^4 y\left[y - \frac{y^4}{64}\right] dy$$

$$2\pi \int_0^4 y^2 - \frac{y^5}{64}\, dy$$

(1)

$$2\pi \left[\frac{y^3}{3} - \frac{y^6}{64 \cdot 6}\right]_0^4$$

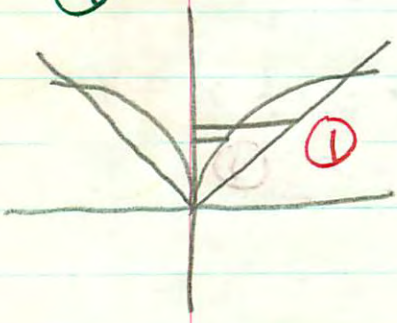
$$2\pi \left[\frac{y^3}{3}\left[1 - \frac{y^3}{128}\right]\right]_0^4$$

(1)

$$2\pi \cdot \frac{64}{3}\left[1 - \frac{1}{2}\right] = 2\pi\left(\frac{64}{3}\right)\left(\frac{1}{2}\right) = \frac{64\pi}{3}$$

(B) using washers (1)

(4)



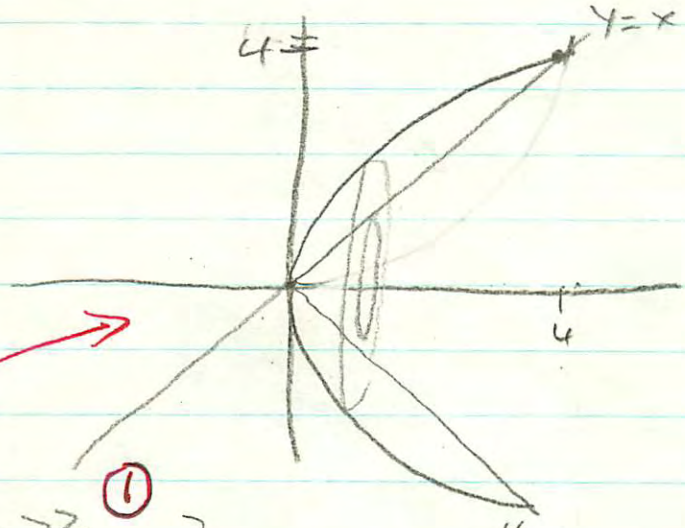
$$\pi \int r_1^2 - r_2^2\, dy$$

$$\pi \int_0^4 \left(y^2 - \frac{y^8}{(64)^2}\right) dy$$

(2)

$4 = 16$
$2^6 \cdot 2^2$
2^8
2^4

$y = (64x)^{\frac{1}{4}}$ $\quad y = x$



A)

(6) $(64x)^{\frac{1}{4}} = x$

$64x = x^4$

$x(x^3 - 64) = 0$

$x(x-4)(x^2+4x+16)$

$\therefore \quad x = 0, \quad x = 4$ ← (2)

(1)

(1) Washers $\quad \pi \int_0^4 \left[\left[(64x)^{\frac{1}{4}} \right]^2 - x^2 \right] dx = \pi \int_0^4 (64x)^{\frac{1}{2}} - x^2 \, dx$

$\pi \int_0^4 8x^{\frac{1}{2}} - x^2 \, dx = \pi \left[\frac{16x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]_0^4$ (1)

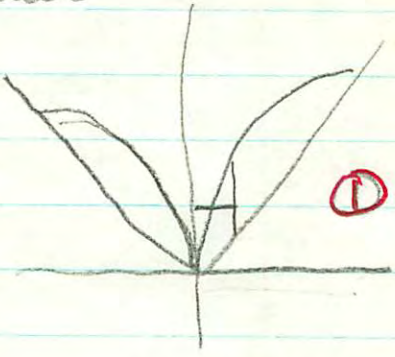
$\pi \left[\frac{16 \cdot 8}{3} - \frac{64}{3} \right] = \frac{8\pi}{3}[16-8] = \frac{8\pi}{3}[8] = \frac{64\pi}{3}$ (1)

R has volume 40π cu units

B) (4)

radius of rev: x $\quad x \Big|_0^4$

height of shell $(64x)^{\frac{1}{4}} - x$ (1)



$2\pi \int_0^4 x \left[(64x)^{\frac{1}{4}} - x \right] dx$

(2)

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(4)

$f(x) = 2\ln(x^2+3) - x \qquad -3 \le x \le 5$

(5) A) $f'(x) = \dfrac{2(2x)}{x^2+3} - 1$ (1)

$= \dfrac{4x - (x^2+3)}{x^2+3} = -\left(\dfrac{x^2-4x+3}{x^2+3}\right)$

(1) $= \dfrac{-(x-3)(x-1)}{x^2+3} \overset{\text{Set}}{=} 0 \qquad x \in \{1, 3\}$

(1)

f'			$-$	0	$+$	0	$-$		
	-3		0	1		3		5	

only asks for abscissa values

(3) C)

$f(-3) = 2\ln 9 + 3 = 4\ln 3 + 3$

↑ max, decrease fn it

$f(3) = 2\ln 9 - 3 = 4\ln 3 - 3$

↑ max f' test

compare maxs

$4\ln 3 + 3 \ge 4\ln 3 - 3$

∴ Abs max occurs at $4\ln 3 + 3$ is the largest of all maxs ∴ Abse max of fctn.

(2)

$x = -3$	max fctn decreases from it
$x = 1$	min f' test
$x = 3$	max f' test
$x = 5$	min fctn decreases to it

(3)

B) f'' Set 0 $\qquad f' = \dfrac{4x}{x^2+3} - 1$ (u, v)

(2) $f'' = \dfrac{(x^2+3)4 - 4x(2x)}{(x^2+3)^2} \overset{\text{Set}}{=} 0$ (1)

$4x^2 + 12 - 8x^2$

$-4x^2 + 12$

$-4(x^2-3)$

$x = \pm\sqrt{3}$

Points of inflection occur at $x = \sqrt{3}$ and $x = -\sqrt{3}$ (1)

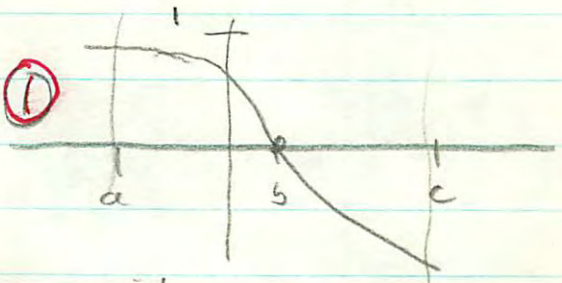
1987
(6)

$f(x) < 1$, $f'(x) < 0 \ \forall x$

(3) A) $f(b) = 0 \quad a < b < c$

$$A = \int_a^b f(x)\,dx + \int_b^c -f(x)\,dx$$

(2) area above x-axis; area below x-axis.



(1) once it crosses, curve CANNOT come back up since $f' < 0 \ \forall x$.

B)

$$g(x) = \frac{1}{f(x)-1}$$

(1) (2)

(4) $g'(x) = -[f(x)-1]^{-2} f'(x)$

$$= \frac{-f'(x)}{(f(x)-1)^2}$$

(1) Denom is > 0 since it is squared. Numerator $-(\text{neg}) = +$ is also > 0 $\therefore$ fraction > 0

$g(x)$ is an increasing fctn. since its derivative is $> 0 \ \forall x$ (1)

(3) C) $h'(x) < 0 \quad F(x) = h(f(x))$ Chain rule (1)

$$F'(x) = h'(f(x))\, f'(x)$$

$(-)(-) = +$

Since $h'(x) < 0 \ \forall x$ then $h'(f(x)) < 0$ (1)

$f'(x)$ is given $< 0 \ \forall x$ $\therefore$

$F'(x) > 0 \ \forall x$

$\therefore F'(x) = h(f(x))$ (1) is an increasing fctn.

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

Number of problems—6

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

Notes: (1) In this examination $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. Let f be the function defined by $f(x) = 7 - 15x + 9x^2 - x^3$ for all real numbers x.
   (a) Find the zeros of f.
   (b) Write an equation of the line tangent to the graph of f at $x = 2$.
   (c) Find the x-coordinates of all points of inflection of f. Justify your answer.

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2. Let f be the function given by $f(x) = \dfrac{9x^2 - 36}{x^2 - 9}$.
   (a) Describe the symmetry of the graph of f.
   (b) Write an equation for each vertical and each horizontal asymptote of f.
   (c) Find the intervals on which f is increasing.
   (d) Using the results found in parts (a), (b), and (c), sketch the graph of f on the axes below.

---

3. A particle moves along the x-axis so that at any time $t \geq 1$ its acceleration is given by $a(t) = \dfrac{1}{t}$. At time $t = 1$, the velocity of the particle is $v(1) = -2$ and its position is $x(1) = 4$.
   (a) Find the velocity $v(t)$ for $t \geq 1$.
   (b) Find the position $x(t)$ for $t \geq 1$.
   (c) What is the position of the particle when it is farthest to the left?

---

4. Let f be the function defined as follows.

$$f(x) = \begin{cases} |x - 1| + 2, & \text{for } x < 1 \\ ax^2 + bx, & \text{for } x \geq 1, \text{ where } a \text{ and } b \text{ are constants.} \end{cases}$$

   (a) If $a = 2$ and $b = 3$, is f continuous for all x? Justify your answer.
   (b) Describe all values of a and b for which f is a continuous function.
   (c) For what values of a and b is f both continuous and differentiable?

1986

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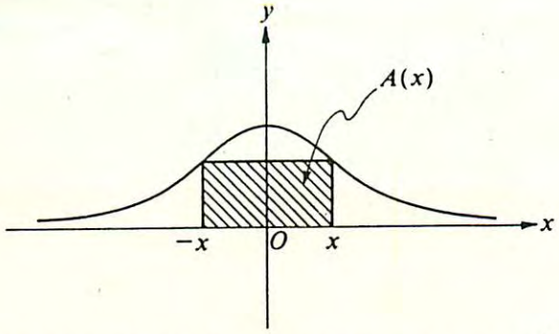
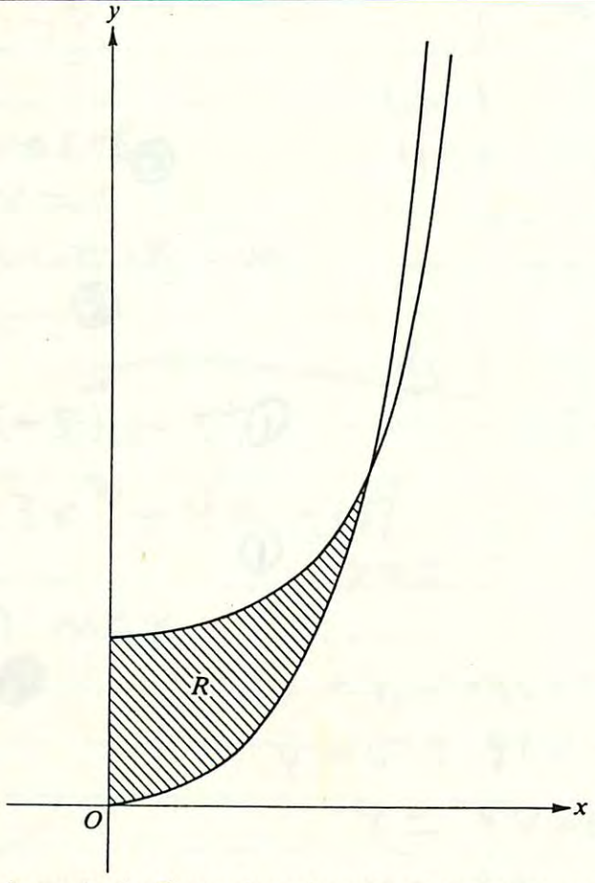


5. Let $A(x)$ be the area of the rectangle inscribed under the curve $y = e^{-2x^2}$ with vertices at $(-x, 0)$ and $(x, 0)$, $x \geq 0$, as shown in the figure above.

   (a) Find $A(1)$.

   (b) What is the greatest value of $A(x)$? Justify your answer.

   (c) What is the average value of $A(x)$ on the interval $0 \leq x \leq 2$?



6. The shaded region R shown in the figure above is enclosed by the graphs of $y = \tan^2 x$, $y = \frac{1}{2}\sec^2 x$, and the y-axis.

   (a) Find the area of region R.

   (b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid formed by revolving region R about the x-axis.

1986

END OF EXAMINATION

① PI occurs when f changes concavity

of inflection occurs at $x = 3$. ②

1986

①

$f(x) = -x^3 + 9x^2 - 15x + 7$
$\quad -(x^3 - 9x^2 + 15x - 7)$

5 A) $x \in \{1, 1, 7\}$

	1	-9	15	-7
1	1	-8	7	0
1	1	-7	0	

$x - 7 = 0$
$x = 7$

The zeros of f(x) are $x = 1$ ②
① → (multiplicity of two)
and $x = 7$ ②

2	1	-7	1	-5

5 B) $f(2) = -(-5) = 5$ ①

$f'(x) = -3x^2 + 18x - 15 \Big|_{x=2} = -12 + 36 - 15 = 9$ ① ①

$(2, 5)$ $m = 9$

$y - y_1 = m(x - x_1)$ ①
$y - 5 = 9(x - 2)$
$y = 9x - 18 + 5$
$y = 9x - 13$ ①

5 c) $f''(x) = -6x + 18$ ①
$= -6(x - 3)$ $\quad f'' =$ zero only at $x = 3$

f'' $\quad + \quad 0 \quad -$
① $\quad 0 \quad 3$

f is concave up for $x < 3$ and concave down after $x = 3$ ∴ only point of inflection occurs at $x = 3$. ②

① PI occur when f changes concavity

$$f(x) = \frac{9x^2-36}{x^2-9} = \frac{9(x-2)(x+2)}{(x-3)(x+3)}$$

(2) A) $f(x)$ is an even fctn since $f(-x) = f(x)$ therefore (1)
f is reflected through the y-axis. (1)

(5) B) Vertical asymptotes occur at non reducible Domain restrictions. (1)
Domain $\stackrel{\text{set}}{=} 0$ & solved $x \in \{\pm 3\}$
Vertical asymptotes $x=3$ and $x=-3$ (1)

Horizontal asymptotes occur when (1)
$\lim_{x\to\pm\infty} f(x) = L$ and are $y = L$.

(1) $$\lim_{x\to\pm\infty} \frac{9x^2-36}{x^2-9} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\pm\infty} \frac{9-\frac{36}{x^2}}{1-\frac{9}{x^2}} = 9$$

only horizontal asymptote is $y=9$ (1)

(5) C) $f(x) = \frac{9x^2-36}{x^2-9}$ u v

$$f'(x) = \frac{(x^2-9)18x - (9x^2-36)2x}{(x^2-9)^2} \quad (1)$$

$$= \frac{18x^3 - 162x - 18x^3 + 72x}{(x^2-9)^2}$$

(1) $$= \frac{-90x}{(x^2-9)^2}$$

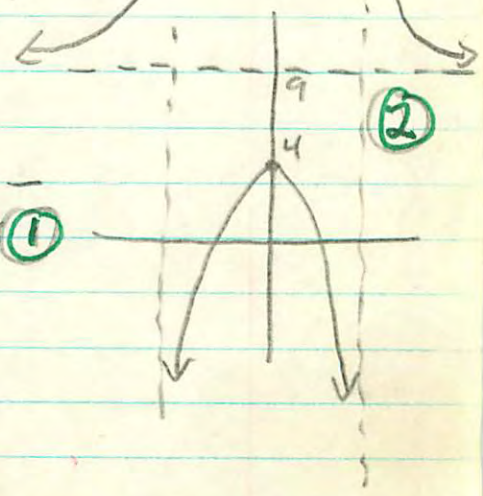
f': $+$ ⊗ $+$ 0 $-$ ⊗ $-$ (1)
-3 0 3

f is increasing $\forall\, x < 0$, $x \neq -3$ (1) (1)

(3) D) $f(0) = 4$

(2) $\lim_{x\to\pm 3^-} f(x) = \frac{+}{-} \Rightarrow -\infty$

$\lim_{x\to\pm 3^+} f(x) = \frac{+}{+} \Rightarrow +\infty$



1986

3

$a(\tau) = \frac{1}{\tau}$ $\quad T = 1 \quad v(1) = -2 \quad x(1) = 4$

5 a) $dv = \int a(\tau)\,d\tau$ 1 $\quad \tau \geq 1 \quad \therefore$ No ∥ bars needed 1

$= \int \frac{1}{\tau}\,d\tau$

1

$v(t) = \ln\tau + c =$

$\ln(1) + c \overset{\text{set}}{=} -2$ 1

$c = -2$

$v(t) = \ln\tau - 2$ 1

6 b) $x(\tau) = \int v(t)\,dt$ 1

$= \int \ln\tau - 2\,dt$

1

$x(\tau) = \tau\ln\tau - \tau - 2\tau + c$

$x(1) = 1 \cdot 0 - 3 + c \overset{\text{set}}{=} 4$ 1

$c = 7$

$\therefore x(\tau) = \tau\ln\tau - 3\tau + 7$ 1

ck $\overset{u}{\tau}\overset{v}{\ln\tau} - \tau$

$\ln\tau + \frac{\tau}{\tau} - 1$

$\ln\tau + 1 - 1$

$\ln\tau$

2

OR

using parts

$u = \ln\tau \quad v = \tau$

$du = \frac{1}{\tau}\,d\tau \quad dv = d\tau$

$\tau\ln\tau - \int d\tau$

$\tau\ln\tau - \tau$

4 c) farthest left is min of $x(\tau)$ 1

$x'(\tau) = v(\tau) = \ln\tau - 2 \overset{\text{set}}{=} 0$

$\ln\tau = 2$

$\tau = e^2$ 1

$x''(\tau) = a(\tau) = \left.\frac{1}{\tau}\right|_{e^2} = \frac{1}{e^2} > 0 \quad \therefore$ minimum f'' test 1

$x(e^2) = e^2\ln e^2 - 3e^2 + 7$

$= 2e^2 - 3e^2 + 7$

$= 7 - e^2$

farthest left $x(T)$ gets is $7 - e^2$ 1

1986
(4)

$$f(x)=\begin{cases} |x-1|+2 & x<1 \\ ax^2+bx & x\geq 1 \end{cases}$$

(6) a) $a=2$ $b=3$. if f is continuous then $\lim_{x\to a} f(x)=f(a)$ (1)

(1) f is continuous $\forall x\neq 1$ because f is a polynomial and all polynomials are continuous. the only discontinuity occurs at $x=1$

(1) $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} |x-1|+2=2$ (1) since $2\neq 5$ limit DNE

(1) $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 2x^2+3x=5$ since limit must exist for f to be cont f is Not continuous at $x=1$. (1)

(3) B) f is cont $\forall a,b$ such that

(2) $\lim_{x\to 1^+} ax^2+bx=2$ ∴

$a+b=2$. f will be continuous when a, b satisfy the relationship $a+b=2$ (1)

(6) c) f is diff.

(1) $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ · def of der.

(L) $\lim_{x\to 1^-} \frac{|x-1|+2-2}{x-1}$

(1) $\lim_{x\to 1^-} \frac{|x-1|}{x-1}$

$\lim_{x\to 1^-} \frac{-(x-1)}{x-1}$

$\lim_{x\to 1^-} -1=-1$

(R) $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$ (1)

$\lim_{x\to 1^+} \frac{ax^2+bx-(a+b)}{x-1}$

$\lim_{x\to 1^+} \frac{a(x^2-1)+b(x-1)}{x-1}$

$\lim_{x\to 1^+} a(x+1)+b=2a+b$

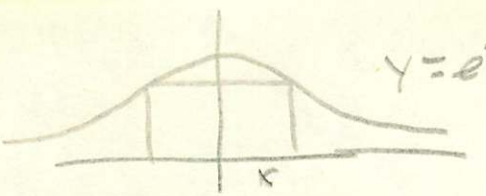
(1) $2a+b$ set $=-1$

$a+(a+b)=-1$

$a+2=-1$

(1) → $a=-3$ ∴ $b=5$ (since $a+b=2$)

f is cont and differentiable when $a=-3$ and $b=5$ (1)



$y = e^{-2x^2}$ 1986 (5)

(4) A) Rectangle $A = bh$ (1)

$= 2x(e^{-2x^2})$

$= 2xe^{-2x^2}$ (2)

$A(1) = 2e^{-2}$ (1)

(5) B) $A(x) = \overset{u}{2x}\overset{v}{e^{-2x^2}}$

$A'(x) = 2e^{-2x^2} + 2x\left[-4xe^{-2x^2}\right]$

(1) $= 2e^{-2x^2} - 8x^2e^{-2x^2}$

$= 2e^{-2x^2}\left[1 - 4x^2\right]$ critical values

$\uparrow$ never zero $\quad (1-2x)(1+2x)$ $\quad x \in \{\pm\frac{1}{2}\}$

f': $-$ 0 $+$ 0 $-$ (1)

$-\frac{1}{2}$ $\quad 0 \quad$ $\frac{1}{2}$

(1) max by f' test at $x = \frac{1}{2}$

$A(\frac{1}{2}) = 2 \cdot \frac{1}{2} e^{-2(\frac{1}{4})}$

(1) $= e^{-\frac{1}{2}}$ is the max area of $A(x)$ (1)

(6) C) average value $\frac{1}{b-a}\int_a^b f(x)\,dx$ (1)

$A(x) = 2xe^{-2x^2}$

$\frac{1}{\cancel{2}-0}\int_0^2 \cancel{2}xe^{-2x^2}\,dx$ (1) let $u = -2x^2$ (1)

$du = -4x\,dx$

$-\frac{1}{4}du = x\,dx$

$-\frac{1}{4}\int_0^{-8} e^u\,du = \frac{1}{4}\int_{-8}^{0} e^u\,du = \frac{1}{4}e^u\Big|_{-8}^{0}$ $\quad x=0 \quad u=0$ (1)

$x=2 \quad u=-8$

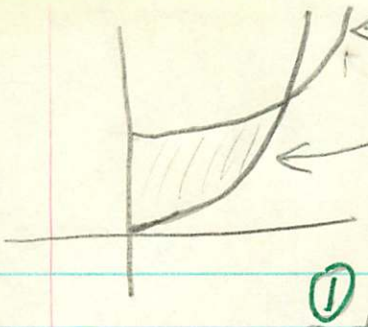
$\frac{1}{4}\left[1 - e^{-8}\right]$ = average value of $A(x)$. (1) (1)

OR

$\frac{1}{2}\int_0^2 \underset{du}{2x}\,\underset{e^{du}}{e^{-2x^2}}\,dx = \frac{1}{2}\left(-\frac{e^{-2x^2}}{2}\right)\Big|_0^2 = \frac{1}{2}\left[-\frac{e^{-8}}{2} + \frac{1}{2}\right]$

$\frac{1}{4}\left[1 - e^{-8}\right]$

(6)



$y = \tan^2 x$

$y = \frac{1}{2}\sec^2 x$

$y = \tan^2 0 = 0$ (2)

$y = \frac{1}{2}\sec^2 0 = \frac{1}{2}$

must show which fcn is which

(10) A) $\tan^2 x \overset{\text{(1) Set}}{=} \frac{1}{2}\sec^2 x$

$\sec^2 x = \tan^2 x + 1$

$\tan^2 x = \frac{1}{2}(\tan^2 x + 1)$

$\frac{1}{2}\tan^2 x = \frac{1}{2}$

$\tan^2 x = 1$

$\tan x = \pm 1$ (1)

$x = \pm \frac{\pi}{4}, \frac{5\pi}{4}$ etc.

(2) for given picture $x = \frac{\pi}{4}$ is intersection point.

$$\int_0^{\frac{\pi}{4}} \frac{1}{2}\sec^2 x - \tan^2 x \, dx \quad \text{(1)}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^2 x \, dx$$

$$\frac{1}{2}\tan x \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^2 x - 1 \, dx \quad \text{(1)}$$

$$- \tan x \Big|_0^{\frac{\pi}{4}} + x \Big|_0^{\frac{\pi}{4}}$$

(1) $\frac{1}{2}\tan x \Big|_0^{\frac{\pi}{4}} + \frac{\pi}{4}$

$-\frac{1}{2} - 0 + \frac{\pi}{4} = \frac{\pi - 2}{4}$ (1)

OR

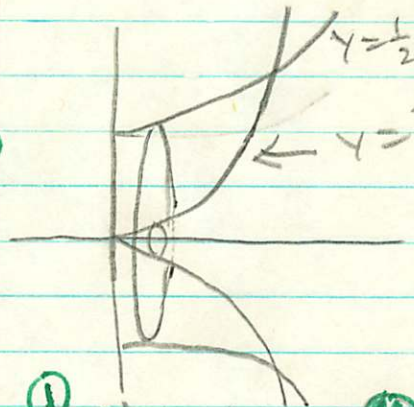
$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\sec^2 x - (\sec^2 x - 1) \, dx$$

$$\int_0^{\frac{\pi}{4}} -\frac{1}{2}\sec^2 x + 1 \, dx$$

$$-\frac{1}{2}\tan x + x \Big|_0^{\frac{\pi}{4}} =$$

$$-\frac{1}{2} + \frac{\pi}{4} - 0 = \frac{\pi - 2}{4}$$

(15) (5) B)



$y = \frac{1}{2}\sec^2 x$

$y = \tan^2 x$

(1) use "washers" (4)

$$\pi \int_0^{\frac{\pi}{4}} \frac{1}{4}\sec^4 x - \tan^4 x \, dx$$

(2) (2)

1985

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

Number of problems—6

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

Notes: (1) In this examination $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. Let f be the function given by $f(x) = \dfrac{2x-5}{x^2-4}$.

   (a) Find the domain of f.

   (b) Write an equation for each vertical and each horizontal asymptote for the graph of f.

   (c) Find $f'(x)$.

   (d) Write an equation for the line tangent to the graph of f at the point $(0, f(0))$.

---

2. A particle moves along the x-axis with acceleration given by $a(t) = \cos t$ for $t \geq 0$. At $t = 0$ the velocity $v(t)$ of the particle is 2 and the position $x(t)$ is 5.

   (a) Write an expression for the velocity $v(t)$ of the particle.

   (b) Write an expression for the position $x(t)$.

   (c) For what values of t is the particle moving to the right? Justify your answer.

   (d) Find the total distance traveled by the particle from $t = 0$ to $t = \dfrac{\pi}{2}$.

---

3. Let R be the region enclosed by the graphs of $y = e^{-x}$, $y = e^{x}$, and $x = \ln 4$.

   (a) Find the area of R by setting up and evaluating a definite integral.

   (b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume generated when the region R is revolved about the x-axis.

   (c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume generated when the region R is revolved about the y-axis.

GO ON TO THE NEXT PAGE

FORM 3HBP

1985

The College Board

Advanced Placement Examination

CALCULUS AB

SECTION II

This insert may be used for reference and/or scratchwork as you answer the free-response questions, but be sure to show all your work on problems and write your answers in the pink booklet. No credit will be given for work shown on this green insert.

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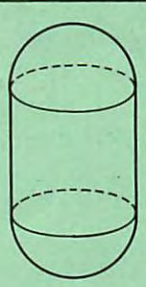
4. Let $f(x) = 14\pi x^2$ and $g(x) = k^2 \sin\left(\frac{\pi x}{2k}\right)$ for $k > 0$.

   (a) Find the average value of f on $[1, 4]$.

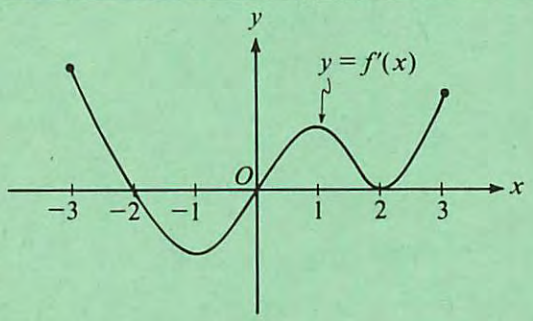
   (b) For what value of k will the average value of g on $[0, k]$ be equal to the average value of f on $[1, 4]$?



5. The balloon shown above is in the shape of a cylinder with hemispherical ends of the same radius as that of the cylinder. The balloon is being inflated at the rate of 261π cubic centimeters per minute. At the instant the radius of the cylinder is 3 centimeters, the volume of the balloon is 144π cubic centimeters and the radius of the cylinder is increasing at the rate of 2 centimeters per minute. (The volume of a cylinder with radius r and height h is $\pi r^2 h$, and the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.)

   (a) At this instant, what is the height of the cylinder?

   (b) At this instant, how fast is the height of the cylinder increasing?



Note: This is the graph of the derivative of f, not the graph of f.

6. The figure above shows the graph of f', the derivative of a function f. The domain of the function f is the set of all x such that $-3 \leq x \leq 3$.

   (a) For what values of x, $-3 < x < 3$, does f have a relative maximum? A relative minimum? Justify your answer.

   (b) For what values of x is the graph of f concave up? Justify your answer.

   (c) Use the information found in parts (a) and (b) and the fact that $f(-3) = 0$ to sketch a possible graph of f on the axes provided below.

END OF EXAMINATION

fg

$$x = \frac{2x-5}{y^2-4}$$

$$x = \frac{2y-5}{y^2-4}$$

$$xy^2 - 4x = 2y - 5$$

$$xy^2 - 2y = 4x - 5$$

$$y(x$$

1985
(1)

$f(x) = \frac{2x-5}{x^2-4}$ $\quad u \quad v$

(3) A) D_f Denom $\overset{\text{set}}{=} 0$ & solved. (1)
$x \neq \{\pm 2\}$ (2)

(4) B) Vertical asympotes at each non reducable (1)
Domain restriction. ie $x=2$, $x=-2$ (1)

horizontal asympotes take $\lim_{x\to\pm\infty}$ (1)

$\lim_{x\to\pm\infty} \frac{2x-5}{x^2-4} = \frac{\infty}{\infty}$ l'hopital

$\lim_{x\to\pm\infty} \frac{2}{x} \Rightarrow 0$ $\therefore$ horizontal asympote at $y=0$ (1)

(5) C) $f' = \frac{(x^2-4)2 - (2x-5)2x}{(x^2-4)^2}$ (1) $\quad \frac{v\,du - u\,dv}{v^2}$ (1)

$= \frac{2x^2 - 8 - 4x^2 + 10x}{(x^2-4)^2} = \frac{-(2x^2-10x+8)}{(x^2-4)^2}$ (2)

$= \frac{-2(x^2-5x+4)}{(x+2)^2(x-2)^2} = \frac{-2(x-1)(x-4)}{(x+2)^2(x-2)^2} = f'(x)$ (1) to see if reducable

(3) D) $f(0) = \frac{-5}{-4} = +\frac{5}{4}$ (1) $\quad f'(0) = \frac{-(8)}{16} = -\frac{1}{2}$ (1)

$y - \frac{5}{4} = -\frac{1}{2}(x-0) \Rightarrow y = \frac{1}{2}x + \frac{5}{4}$. (1)

1985
(2)

$a(t) = \cos t \quad t \geq 0 \quad v(0) = 2 \quad x(0) = 5$

(4) a) $\frac{dv}{dt} = a(t) = \cos t$

$\int dv = \int \cos t \, dt$ (1)

$v = \sin t + C$

$v(0) = \sin 0 + C \overset{\text{set}}{=} 2 \therefore C = 2$ (1)

$v(t) = \sin t + 2$ (1)

(4) B) $\frac{dx(t)}{dt} = v(t) = \sin t + 2$

$\int dx = \int \sin t + 2 \, dt$ (1)

$x(t) = -\cos t + 2t + C$ (1)

$x(0) = -\cos(0) + 2 \cdot 0 + C \overset{\text{set}}{=} 5$

$-1 + 0 + C = 5$

$C = 6$ (1)

$\therefore x(t) = -\cos t + 2t + 6$ (1)

(3) c) Particle moves to right when $v(t) > 0$ (1)

$\sin t + 2 > 0 \quad \forall t > 0$ since ranged $-1 \leq \sin t \leq 1$ $\sin t + 2$ is always > 0, $\therefore$ particle moves constantly to the right. (2)

(4) D) Since particle moves constantly to right (Pt c), (2) total distance is Difference between $x(\frac{\pi}{2})$ and $x(0)$

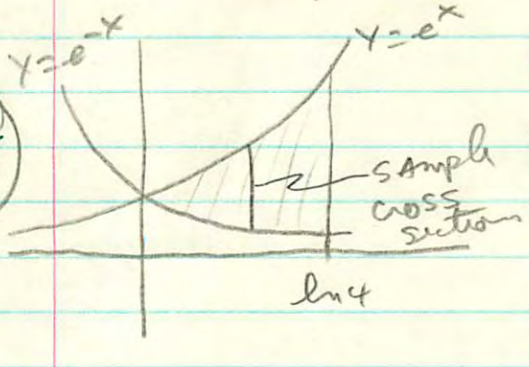
$x(\frac{\pi}{2}) - x(0) = -\cos(\frac{\pi}{2}) + 2(\frac{\pi}{2}) + 6 - [-\cos(0) + 2 \cdot 0 + 6]$

$-0 + \pi + 1 = \pi + 1$ (2)

1985
(3)

$y = e^{-x}$, $y = e^{x}$ $x = \ln 4$

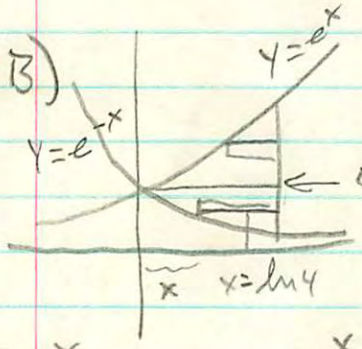
Picture 3pts



(7) A) $\int_0^{\ln 4} e^x - e^{-x}\,dx = e^x + e^{-x}\Big|_0^{\ln 4} =$ (2) (1)

$4 + \frac{1}{4} - 1 - 1 = 2\frac{1}{4} = \frac{9}{4}$ (1) sq units

(4) B)



using shells

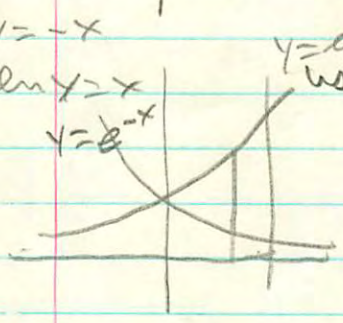
$2\pi\int$ rad of rev · length dy

changes at $y=1$

$2\pi\left[\int_{\frac{1}{4}}^{1} y(\ln 4 - (-\ln y))\,dy + \int_1^4 y(\ln 4 - \ln y)\,dy\right]$

$\ln y = -x$

$-\ln y = x$



using washers

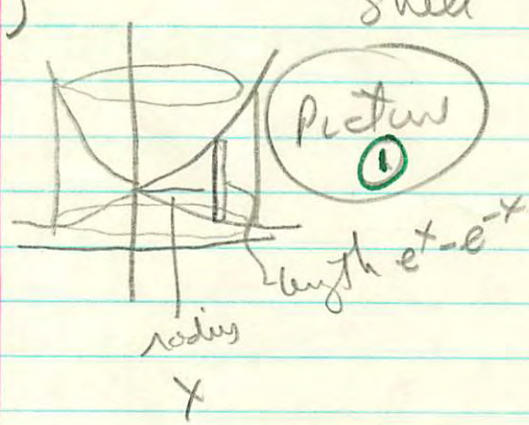
$\pi\int r^2\,dx$ (1)

Picture (1)

$\pi\int_0^{\ln 4} e^{2x} - e^{-2x}\,dx$ (2)

(4) c)

Shell



Picture (1)

$2\pi\int_0^{\ln 4} x(e^x - e^{-x})\,dx$ (2)

(1) $2\pi\int$ radius height dx

1985
(4)

$f(x) = 14\pi x^2$ $g(x) = k^2 \sin\left(\frac{\pi x}{2k}\right)$ $k > 0$

64
14.
256
64
896

(5) a) average value (1) $\frac{1}{\text{length}} \int_a^b f(x)\,dx$

$\frac{1}{4-1} \int_1^4 14\pi x^2\,dx$ (1) $= \frac{1}{3} \cdot \frac{14\pi x^3}{3} \Big|_1^4 = \frac{896\pi}{9} - \frac{14\pi}{9}$ (2)

$\frac{882\pi}{9} = 98\pi$

(10) b) $\frac{1}{k} \int_0^k k^2 \sin\frac{\pi x}{2k}\,dx$ (1) $= k \int_0^k \sin\frac{\pi x}{2k}\,dx$

$k\left(-\frac{2k}{\pi}\right) \cos\frac{\pi x}{2k} \Big|_0^k$ (2) $= -\frac{2k^2}{\pi} \cos\frac{\pi x}{2k} \Big|_0^k = \frac{-2k^2}{\pi}\left(\cos\frac{\pi}{2} - \cos 0\right)$ (2)

$0 - 1$

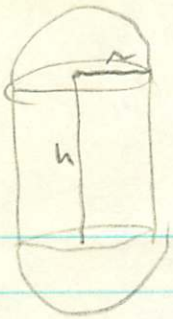
one is $\frac{2k^2}{\pi}$ (2) set (1) $= 98\pi$

$\therefore k^2 = 49\pi^2$

$k = \pm 7\pi$ (1) $k > 0$ Given;

$k = 7\pi$. (1)

1985
(5)



$\frac{dV}{dt} = 261\pi$

$V = \pi r^2 h + \frac{4}{3}\pi r^3$

Recognition (1)

(5) 4) $r = 3$ Given

$144\pi = V$ Given

(2) $144\pi = 9\pi h + \frac{4}{3}\pi \cdot 27$

$108\pi = 9\pi h$

$12 = h$ (2) height is 12 at that instant

(10) B) $\frac{dh}{dt} = ?$

(3) $\frac{dr}{dt} = 2$

$r = 3$

$h = 12$

$\frac{dV}{dt} = 261\pi$

$V = \pi r^2 h + \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt} + 4\pi r^2 \frac{dr}{dt}$ (5)

(1) $261\pi = 2\pi(3)(12)(2) + \pi(3)^2 \frac{dh}{dt} + 4\pi(3)^2 \cdot 2$

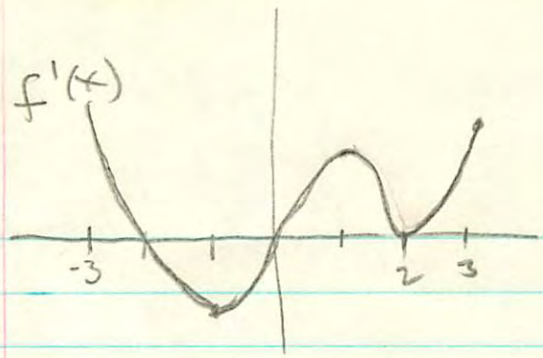
$261 = 144 + 9\frac{dh}{dt} + 72$

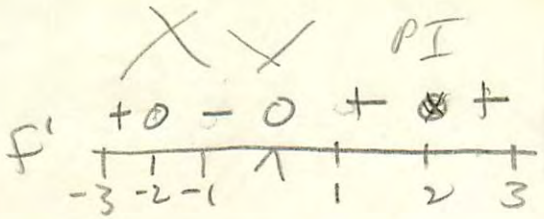
$261 = 216 + 9\frac{dh}{dt}$

$45 = 9\frac{dh}{dt}$

$\frac{dh}{dt} = 5$ (1) (1) the height is increasing at 5 c/minute.

1985
(6)

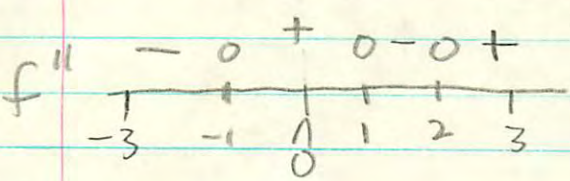




(5) A) critical values of x are (1) zeros of f': $x \in \{-2, 0, 2\}$.

(1) $(-2, f(-2))$ is a rel max by F' test ie $x = -2$
ENDpts (1) $(3, f(3))$ is a rel max $f' > 0$ ∴ f increases to that point, $x = 3$
(1) $(0, f(0))$ is a rel min by f' test ie $x = 0$
ENDpts (1) $(-3, f(-3))$ is a rel min $f' > 0$ ∴ f increases from that point $x = -3$
Note $(2, f(2))$ is a Lazy pt of inflection
Neither a max nor a min.

(6) B) EXTremums in f' are zeros of f''
f'' is the slope of f' tangent
thus when f' is decreasing, f'' is Neg.
when f' is increasing f'' > 0

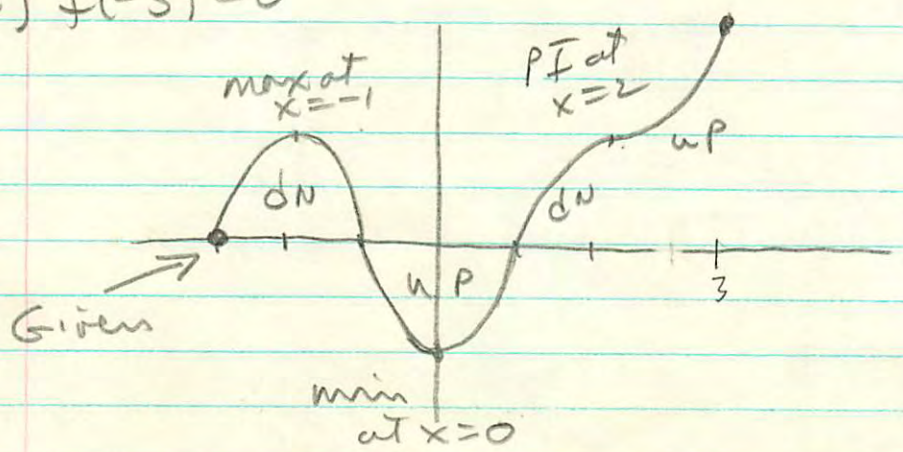


(3)

(2) f is concave up for $-1 < x < 1$ and $2 < x \leq 3$

f is concave up when $f'' > 0$ (1)
f is concave dn when $f'' < 0$
f is neither when $f'' = 0$.

(4) c) $f(-3) = 0$



FORM 3GBP

1984

The College Board

Advanced Placement Examination

CALCULUS AB

SECTION II

Time—1 hour and 15 minutes

Number of problems—5

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS. IF YOU DO NOT EXPRESS A NUMERICAL ANSWER IN EXACT FORM, FOR EXAMPLE, IN TERMS OF RADICALS, e, OR π, THEN YOU SHOULD EXPRESS THAT ANSWER TO THREE SIGNIFICANT DIGITS. CALCULATORS MAY BE USED ON THIS EXAMINATION.

Notes: (1) In this examination, $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. A particle moves along the X-axis so that, at any time $t \geq 0$, its acceleration is given by $a(t) = 6t + 6$. At time $t = 0$, the velocity of the particle is -9, and its position is -27.

   (a) Find $v(t)$, the velocity of the particle at any time $t \geq 0$.

   (b) For what values of $t \geq 0$ is the particle moving to the right?

   (c) Find $x(t)$, the position of the particle at any time $t \geq 0$.

---

2. Let f be the function defined by $f(x) = \dfrac{x + \sin x}{\cos x}$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

   (a) State whether f is an even function or an odd function. Justify your answer.

   (b) Find $f'(x)$.

   (c) Write an equation of the line tangent to the graph of f at the point where $x = 0$.

---

3. Let R be the region enclosed by the X-axis, the Y-axis, the line $x = 2$, and the curve $y = 2e^x + 3x$.

   (a) Find the area of R by setting up and evaluating a definite integral. Your work must include an antiderivative.

   (b) Find the volume of the solid generated by revolving R about the Y-axis by setting up and evaluating a definite integral. Your work must include an antiderivative.

TURN THE PAGE

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4. A function f is continuous on the closed interval $[-3, 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive	Fails to exist	Negative	0	Negative
$f''(x)$	Positive	Fails to exist	Positive	0	Negative

(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.

(b) What are the x-coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.

(c) On the axes provided, sketch a graph that satisfies the given properties of f.

---

5. The volume V of a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$ is increasing at the rate of 28π cubic units per second. At the instant when the radius r of the cone is 3 units, its volume is 12π cubic units and the radius is increasing at $\frac{1}{2}$ unit per second.

(a) At the instant when the radius of the cone is 3 units, what is the rate of change of the area of its base?

(b) At the instant when the radius of the cone is 3 units, what is the rate of change of its height h?

(c) At the instant when the radius of the cone is 3 units, what is the instantaneous rate of change of the area of its base with respect to its height h?

END OF EXAMINATION

1 A) $A(t) = 6t + 6$ $\int a(t)\,dt = v(t) + C$ (1)

7 $\int 6t + 6\,dt = 3t^2 + 6t + C = v(t)$ (1)

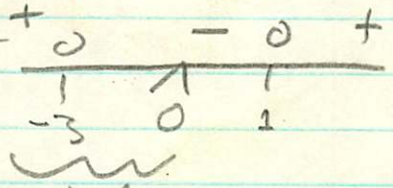
$v(t)\big|_{t=0} = -9 \overset{\text{set}}{=} 3t^2 + 6t + C \;\therefore\; C = -9$ (2) (1)

$\therefore\; v(t) = 3t^2 + 6t - 9$ (2)

6 B) $v(t) = 3t^2 + 6t - 9 \overset{\text{set}}{=} 0$ (2) (1)

$t^2 + 2t - 3 = 0$

$(t+3)(t-1) = 0$



(1) (1) Conclusion based on work.
The particle moved to the right $\forall t > 1$

(2) reject, domain restriction.

7 C) $x(t) = \int v(t)\,dt = x(t) + C$ (1)

$x(t) = \int 3t^2 + 6t - 9\,dt$ (1)

$x(t)\big|_{t=0} = t^3 + 3t^2 - 9t + C \Rightarrow C = -27$ (2) (1)

$x = -27$

the Position fctn: $x(t) = t^3 + 3t^2 - 9t - 27$ (2)

1984

(2)

(12) (A) $f(x) = \dfrac{x + \sin x}{\cos x}$ $-\frac{\pi}{2} < x < \frac{\pi}{2}$ stops $\cos x$ from being zero. 4th & 1st quads.

(1) f is odd if $f(-x) = -f(x)$

(1) f is even if $f(-x) = f(x)$

(2) $f(-x) = \dfrac{-x + \sin(-x)}{\cos(-x)}$. sin is odd and cos is even

(3) $f(-x) = \dfrac{-x - \sin x}{\cos x}$

(1) $= -\left(\dfrac{x + \sin x}{\cos x}\right)$

(2) $= -f(x)$ ∴ f is an odd fctn. (2)

(5) B) $f(x) = \dfrac{x + \sin x}{\cos x}$ $\frac{u}{v}$ (1) $\dfrac{v\,du - u\,dv}{v^2}$

$f'(x) = \dfrac{\cos x(1 + \cos x) - (x + \sin x)(-\sin x)}{\cos^2 x}$ (2)

$= \dfrac{\cos x + \cos^2 x + x\sin x + \sin^2 x}{\cos^2 x}$

$f'(x) = \dfrac{1 + \cos x + x\sin x}{\cos^2 x}$ (2)

(3) C) $f(0) = \dfrac{0+0}{1} = 0$ ∴ pt is $(0, 0)$ (1)

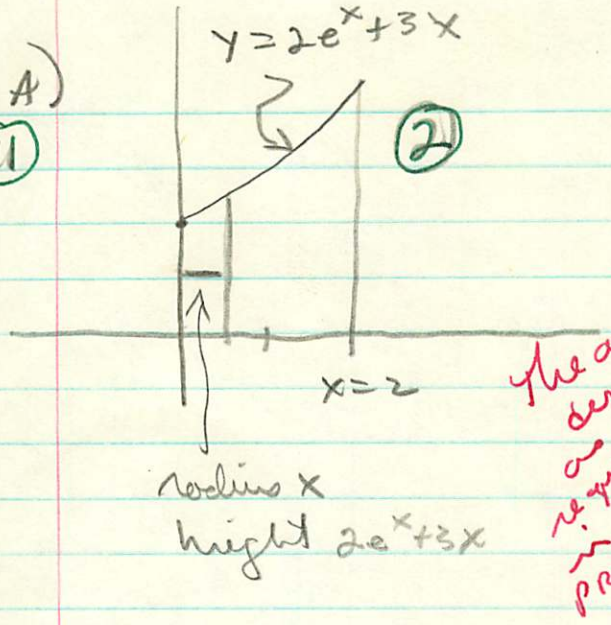
$f'(0) = \dfrac{1 + 1 + 0}{1} = 2 = m$ (1)

∴ equation of tangent line is $y = 2x$. (1)

1984

(3)

A) (11)



(1) $\int_0^2 y\,dx$

(2) $\int_0^2 2e^x + 3x\,dx =$ (1)

The anti derivative we need in problem

(2) $2e^x + \frac{3}{2}x^2 \Big|_0^2 = 2e^2 + 6 - 2 =$ (2)

Area under curve $= 2e^2 + 4$ (1)

(9) B) using shells (1)

(1) $2\pi \int$ (radius) height

(2) $2\pi \int_0^2 x(2e^x + 3x)\,dx = 2\pi \int_0^2 2xe^x + 3x^2\,dx$

(1) $u = 2x \quad v = e^x$

$du = 2\,dx \quad dv = e^x dx$

$= 2\pi \left[2xe^x \Big|_0^2 - 2\int e^x dx + x^3 \Big|_0^2 \right]$

(2) $= 2\pi \left[2xe^x - 2e^x + x^3 \right]_0^2 =$

(1) $2\pi \left[4e^2 - 2e^2 + 8 - 0 + 2 - 0 \right] =$

(1) $2\pi \left[2e^2 + 10 \right] = 4\pi e^2 + 20\pi =$ Total Volume.

1984
(4)

f is cont $[-3, 3]$ $f(-3) = 4$ $f(3) = 1$

	f	4			1	
	f'	+	∅	−	∅	−
	f''	+	∅	+	∅	−
		−3		−1	1	3

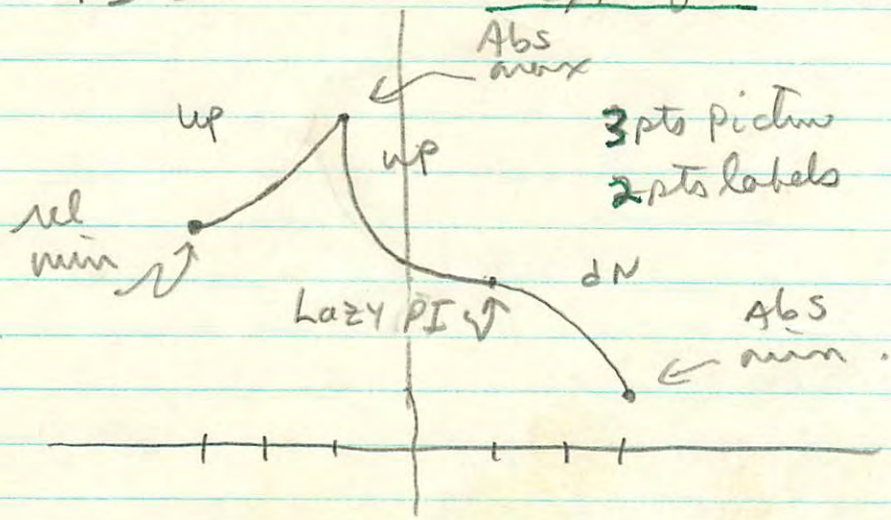
(9) A) (4) at $x = -1$ from left f increases from $y = 4$ then f decreases from $(-1, f(-1))$, this is abs max since f continues to decrease, it is the only max.

(5) compared to other min: $x = 3$ is absolute minimum. f decreases to $(1, f(1))$ but continues to decrease to end of interval. "f decreases" to the point $(3, 1)$ a min. Since it is the lowest of all mins [$(-3, 4)$ is abs min] it is abs.

B) (6) Point of inflection. Pg 212 f'' changes sign.

Only one inflection point – at $x = 1$. This is the only location where the sign of f'' changes, the point at $x = -1$ is a "sharp corner" since the second derivative has the same sign on both sides. This PI is called a "Lazy PT of Inf" since $f'_{abs} = 0$,

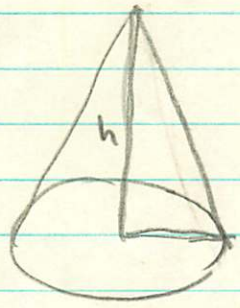
C) (5)



1984 (5)

$V = \frac{1}{3}\pi r^2 h$ $\quad \frac{dV}{dt} = 28\pi$ $\quad r = 3 \Rightarrow V = 12\pi$ $\quad \frac{dr}{dt} = \frac{1}{2}$

A) (7)



(1) $\frac{dA}{dt} = ?$ $\quad A = \pi r^2$ (1)

(1) $\frac{dr}{dt} = \frac{1}{2}$ $\quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$ (2) $= 6\pi \cdot \frac{1}{2} = 3\pi$

$\frac{dr}{dt} = \frac{1}{2}$, $r = 3$

Area of base (a circle) is changing at a rate of 3π sq units / sec.

B) (8)

(1) $\frac{dh}{dt} = ?$ $\quad V = \left(\frac{1}{3}\pi r^2\right)(h)$ (1) (V = u · v)

(3) $\frac{dV}{dt} = h \cdot \frac{2}{3}\pi r \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$

$12\pi = \frac{1}{3}\pi 3^2 h$ (1)

$4 = h$

$28\pi = 4 \cdot \frac{2}{3}\pi \cdot 3 \frac{1}{2} + \frac{1}{3}\pi 9 \frac{dh}{dt}$

$28 = 4 + 3\frac{dh}{dt}$

(2) $8 = \frac{24}{3} = \frac{dh}{dt}$ = rate of change of height, $\frac{\text{linear units}}{\text{sec.}}$

c) $\frac{dA}{dh} = \frac{dA}{dt} \cdot \frac{dt}{dh}$ by chain rule (3)

5

$\therefore \quad \frac{3\pi}{8} = \frac{\frac{dA}{dt}}{\frac{dh}{dt}}$ (2) rate of change of area wrt height

1983
AB

CALCULUS AB

SECTION II

Time—1 hour and 15 minutes

Number of problems—5

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS. IF YOU DO NOT EXPRESS A NUMERICAL ANSWER IN EXACT FORM, FOR EXAMPLE, IN TERMS OF RADICALS, e, OR π, THEN YOU SHOULD EXPRESS THAT ANSWER TO THREE SIGNIFICANT DIGITS. CALCULATORS MAY BE USED ON THIS EXAMINATION.

Notes: (1) In this examination, $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. Let f be the function defined by $f(x) = -2 + \ln(x^2)$.

   (a) For what real numbers x is f defined?

   (b) Find the zeros of f.

   (c) Write an equation for the line tangent to the graph of f at $x = 1$.

---

2. A particle moves along the X-axis so that at time t its position is given by $x(t) = t^3 - 6t^2 + 9t + 11$.

   (a) What is the velocity of the particle at $t = 0$?

   (b) During what time intervals is the particle moving to the left?

   (c) What is the total distance traveled by the particle from $t = 0$ to $t = 2$?

---

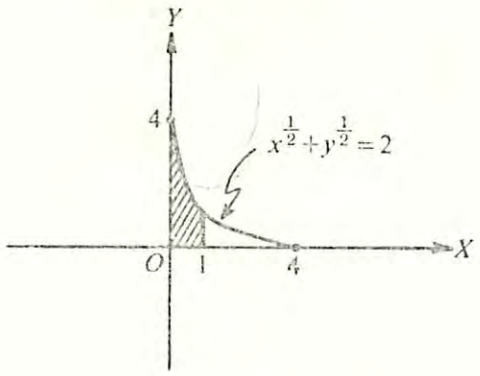
3. Let f be the function defined for $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$ by $f(x) = x + \sin^2 x$.

   (a) Find all values of x for which $f'(x) = 1$.

   (b) Find the x-coordinates of all minimum points of f. Justify your answer.

   (c) Find the x-coordinates of all inflection points of f. Justify your answer.

1983
AB



4. The figure above shows the graph of the equation $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$. Let R be the shaded region between the graph of $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 2$ and the X-axis from $x = 0$ to $x = 1$.

   (a) Find the area of R by setting up and integrating a definite integral.

   (b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid formed by revolving the region R about the X-axis.

   (c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume of the solid formed by revolving the region R about the line $x = 1$.

---

5. At time $t = 0$, a jogger is running at a velocity of 300 meters per minute. The jogger is slowing down with a negative acceleration that is directly proportional to time t. This brings the jogger to a stop in 10 minutes.

   (a) Write an expression for the velocity of the jogger at time t.

   (b) What is the total distance traveled by the jogger in that 10-minute interval?

END OF EXAMINATION

1983
BC

CALCULUS BC

SECTION II

Time—1 hour and 15 minutes

Number of problems—5

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS. IF YOU DO NOT EXPRESS A NUMERICAL ANSWER IN EXACT FORM, FOR EXAMPLE, IN TERMS OF RADICALS, e, OR π, THEN YOU SHOULD EXPRESS THAT ANSWER TO THREE SIGNIFICANT DIGITS. CALCULATORS MAY BE USED ON THIS EXAMINATION.

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1. Let f be the function defined for $\frac{\pi}{6} \leqq x \leqq \frac{5\pi}{6}$ by $f(x) = x + \sin^2 x$.

   (a) Find all values of x for which $f'(x) = 1$.

   (b) Find the x-coordinates of all minimum points of f. Justify your answer.

   (c) Find the x-coordinates of all inflection points of f. Justify your answer.

---

2. Consider the curve $y = 2x^{\frac{1}{2}}$ from $x = 3$ to $x = 8$.

   (a) Set up, but do not integrate, an integral expression in terms of a single variable for the length of the curve.

   (b) Let S be the surface generated by revolving the curve about the X-axis. Find the area of S by setting up and integrating a definite integral.

---

3. At time $t = 0$, a jogger is running at a velocity of 300 meters per minute. The jogger is slowing down with a negative acceleration that is directly proportional to time t. This brings the jogger to a stop in 10 minutes.

   (a) Write an expression for the velocity of the jogger at time t.

   (b) What is the total distance traveled by the jogger in that 10-minute interval?

---

4. Consider the differential equation $\frac{dy}{dx} + 2xy = xe^{(-x^2+x)}$.

   (a) Find the general solution of the differential equation.

   (b) Find the particular solution of the differential equation that satisfies the condition $y = 3$ when $x = 0$.

---

5. Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, where $a_0 = 1$ and $a_n = \left(\frac{7}{n}\right) a_{n-1}$ for $n \geq 1$.

   (a) Find the first four terms and the general term of the series.

   (b) For what values of x does the series converge?

   (c) If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, find the value of $f'(1)$.

END OF EXAMINATION

1983
AB
#1

$f(x) = -2 + \ln(x^2)$

5 A) D_f $x \in \mathbb{R}$, $x \neq 0$ (3) Since x is squared $x<0$ is OK. (2)

8 B) $-2 + \ln(x^2) \overset{\text{set}}{=} 0$ (3)

$2\ln|x| = 2$

$\ln|x| = 1$

$x = \pm e$

$f(x) = 0$ (5)

$\forall x \in \{\pm e\}$

1

7 C) $f(1) = -2 + 0 = -2$ (2) $(1, -2)$

$f'(x) = \frac{2x}{x^2} = \frac{2}{x}\Big| = 2 = m$ (1)

$x=1$

$y + 2 = 2(x-1)$

$y = 2x - 4$ (3)

A) $x \neq 0$ +3 pts

B) +early 3

1983
AB
#2

$x(t) = t^3 - 6t^2 + 9t + 11$

5 A) $v = \frac{d\,x(t)}{dt}$ 2

$v(t) = 3t^2 - 12t + 9 \Big|_{t=0} = 9$ 5

6 B) $v(t) = 3(t^2 - 4t + 3)$ 2
$3(t-3)(t-1)$

2

$+ \quad 0 \quad - \quad 0 \quad +$

$0 \quad 1 \quad 3$

The particle travels to the left for $t \in (1, 3)$ ie 3 $1 < t < 3$

9 C) Sum the distance from $t=0$ to $t=1$ 3
plus the distance from $t=1$ to $t=2$.

The distance from 0 to 2 computed via a
1 single integral would yield the distance from $t=0$; Not the total distance traveled.

Position at $T=$

	1	-6	9	11	
0	1			11	
1	1	-5	4	15	4
2	1	-4	1	13	2
					6

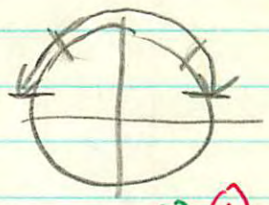
3

Total distance traveled is 6.

2

1983
AB
#3

$f(x) = x + \sin^2 x \qquad \frac{\pi}{6} \le x \le \frac{5\pi}{6}$



6 a) $f'(x) = 1 + 2\sin x \cos x \overset{\text{Set}}{=} 1$ (2) (1)

$\sin x = 0 \quad \cos x = 0$

Reject (1) $\hookrightarrow \Rightarrow x = \frac{\pi}{2}$ (1) (1)

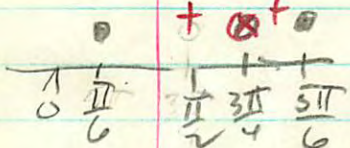
$\sin x \neq 0$ for the given interval (1)

$f'(x) = 1$ when $x = \frac{\pi}{2}$

8 b) $f'(x) = 1 + 2\sin x \cos x \overset{\text{Set}}{=} 0$ (1)

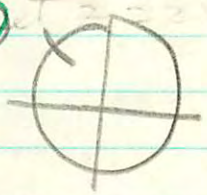
$\sin^2 x + \cos^2 x + 2\sin x \cos x = 0$

$(\sin x + \cos x)^2 = 0$ (2)



$\frac{\pi}{6} \le x \le \frac{5\pi}{6}$

$\pi \le 2x \le 5\pi$



$\sin x = -\cos x$

$x = \frac{3\pi}{4}$ (1)

$f'(x) > 0 \ \forall \ x \neq \frac{3\pi}{4}$ (because it is squared)

$\therefore$ $f(x)$ is increasing from $\frac{\pi}{6} \to \frac{5\pi}{6}$ (2) with a wavy inflection pt at $x = \frac{3\pi}{4}$

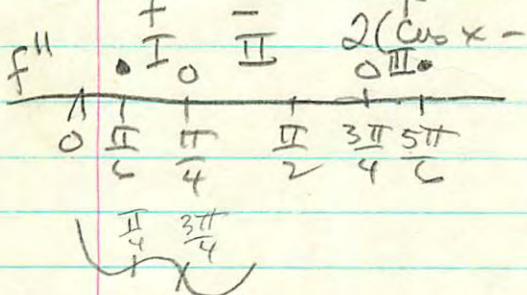
Conclusion: f has only one a minimum at $x = \frac{\pi}{6}$ (2)

6 c) $f'(x) = 1 + 2\sin x \cos x$

$f''(x) = 2\cos^2 x - 2\sin^2 x \overset{\text{Set}}{=} 0$ (2)

$\sin x = \pm \cos x \Rightarrow x \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$

$f'' = 2(\cos x - \sin x)(\cos x + \sin x)$



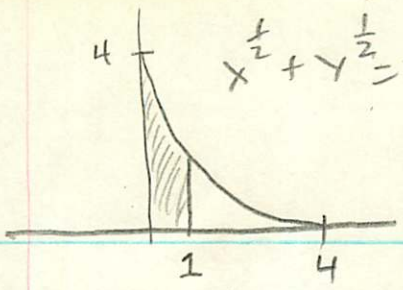
$f' = 2(\cos x - \sin x)(\cos x + \sin x)$

IN I	$\cos > \sin$	+	+	+	+
II	$\sin > \cos x$	+	−	+	−
III	$\lvert\cos\rvert > \sin x$	+	−	−	+

(3)

$f(x)$ has inflection pt at $x \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (1)

1983
AB
#4



7 A) $x^{\frac{1}{2}}+y^{\frac{1}{2}}=2$

$y^{\frac{1}{2}}=2-\sqrt{x}$ (2)

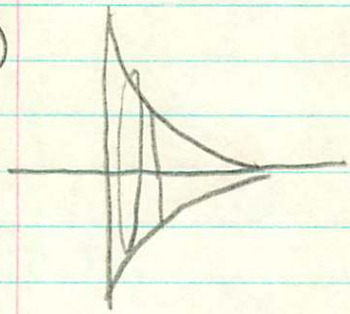
$y=4-4\sqrt{x}+x$

Enclosed area is (1) $\frac{11}{6}$

$\int_0^1 4-4\sqrt{x}+x\,dx = 4x-\frac{8x^{3/2}}{3}+\frac{x^2}{2}\Big|_0^1$ (2)

$4-\frac{8}{3}+\frac{1}{2}=\frac{24+16+3}{6}=\frac{11}{6}$ (2)

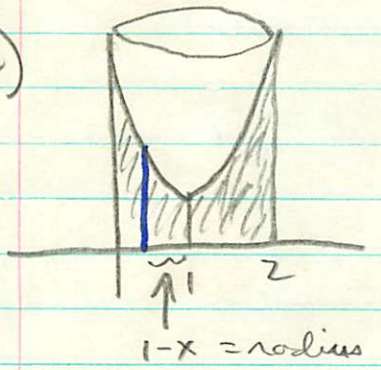
6 B)



(2) $\pi\int r^2\,dx$ $r=y$ from part A.

(4) $\pi\int_0^1 (4-4\sqrt{x}+x)^2\,dx$ (6)

7 C)

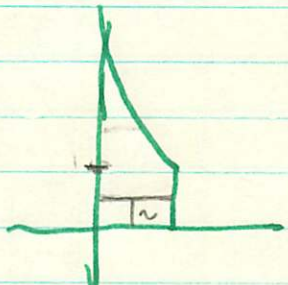


$2\pi\int_0^1 (1-x)(2-\sqrt{x})^2\,dx$ (7)

radius · height of one of cylinder.

using shell technique

#4b using shells



$2\pi\int_3^1 y\,dy+2\pi\int_1^4 y(4-4\sqrt{y}+y)\,dy$

1983
AB
#5

$T=0 \quad v = 300$ meters/sec.

$$\frac{dv}{dt} = f(t) = -kt$$

14 a) $v(0) = 300$
$v(10) = 0$

$$\frac{dv}{dt} = -kt \quad (5)$$

$$\int dv = \int -kt\,dt \quad (1)$$

$$v = -\frac{kt^2}{2} + c \quad (1)$$

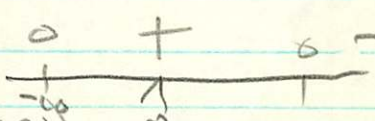
$$v\Big|_{t=0} = 300 \overset{\text{Set}}{=} -\frac{kt^2}{2} + c \quad (1) \quad \therefore \quad c = 300 \quad (1)$$

$$v\Big|_{t=10} = 0 \overset{\text{Set}}{=} -k\frac{100}{2} + 300$$

$$k = 6 \quad (2)$$

$$v = -3t^2 + 300 \quad (3) \qquad \frac{dv}{dt} = -6t$$

b) $\frac{ds}{dt} = -3t^2 + 300 \quad (1) \qquad -3(t^2 - 100)$



$$ds = (-3t^2 + 300)\,dt \quad (1)$$

$$\int_0^{10} ds = S(t) = \int_0^{10} -3t^2 + 300\,dt \quad (1)$$

$$S(t) = -t^3 + 300t\Big|_0^{10} = -1000 + 3000 = 2000 \quad (2)$$

The jogger traveled 2000 meters in the 10 minute interval. (1)

1983
BC

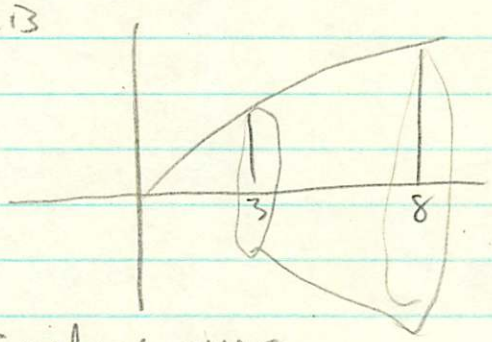
1 Sec 1983 #3

$Y = 2x^{\frac{1}{2}}$

2A $L = \int_3^8 \sqrt{1+\left(\frac{dY}{dx}\right)^2}\,dx = \int_3^8 \sqrt{1+\left(x^{-\frac{1}{2}}\right)^2} =$

$\int_3^8 \sqrt{x^{-1}+1}\,dx = \int_3^8 \sqrt{\frac{1+x}{x}}\,dx$

2B



surface area

$S = \int_3^8 2\pi r\,dx =$

$\int_3^8 4\pi x^{\frac{1}{2}}\,dx = 4\pi\frac{2}{3}x^{\frac{3}{2}}\Big|_3^8 =$

$\frac{8\pi}{3}\left(8^{3/2}-3^{3/2}\right)$

3, Sec 1983 #5

FORM 3EBP

19[illegible]

The College Board

Advanced Placement Examination

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

Number of problems—7

Percent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

Notes: (1) In this examination, ln x denotes the natural logarithm of x (that is, logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which f(x) is a real number.

1. A particle moves along the X-axis in such a way that its acceleration at time t for $t > 0$ is given by $a(t) = \frac{3}{t^2}$. When $t = 1$, the position of the particle is 6 and the velocity is 2.

   (a) Write an equation for the velocity, v(t), of the particle for all $t > 0$.

   (b) Write an equation for the position, x(t), of the particle for all $t > 0$.

   (c) Find the position of the particle when $t = e$.

---

2. Given that f is the function defined by $f(x) = \frac{x^3 - x}{x^3 - 4x}$.

   (a) Find $\lim_{x \to 0} f(x)$.

   (b) Find the zeros of f.

   (c) Write an equation for each vertical and each horizontal asymptote to the graph of f.

   (d) Describe the symmetry of the graph of f.

   (e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.

---

3. Let R be the region in the first quadrant that is enclosed by the graph of $y = \tan x$, the X-axis, and the line $x = \frac{\pi}{3}$.

   (a) Find the area of R.

   (b) Find the volume of the solid formed by revolving R about the X-axis.

[illegible]

This insert may be used for reference and/or scratchwork as you answer the free-response questions, but be sure to show all your work on problems and write your answers in the pink booklet. No credit will be given for work shown on this green insert.

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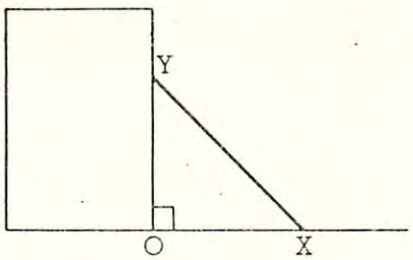
1982



4. A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

   (a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building.

   (b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

---

5. Let f be the function defined by $f(x) = (x^2 + 1)\, e^{-x}$ for $-4 \leq x \leq 4$.

   (a) For what value of x does f reach its absolute maximum? Justify your answer.

   (b) Find the x-coordinates of all points of inflection of f. Justify your answer.

---

6. A tank with a rectangular base and rectangular sides is to be open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs \$10 per square meter for the base and \$5 per square meter for the sides, what is the cost of the least expensive tank?

---

7. For all real numbers x, f is a differentiable function such that $f(-x) = f(x)$. Let $f(p) = 1$ and $f'(p) = 5$ for some $p > 0$.

   (a) Find $f'(-p)$.

   (b) Find $f'(0)$.

   (c) If ℓ_1 and ℓ_2 are lines tangent to the graph of f at $(-p, 1)$ and $(p, 1)$, respectively, and if ℓ_1 and ℓ_2 intersect at point Q, find the x- and y-coordinates of Q in terms of p.

END OF EXAMINATION

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1982 (1)

(1) $a(t) = \frac{3}{t^2}$ $\quad t=1 \quad x(t)=6 \quad v(t)=2$

8 (A) $\frac{dv}{dt} = a(t) = \frac{3}{t^2}$ (1) (1)

$dv = \frac{3}{t^2}\,dt$

(1) $\int dv = \int \frac{3}{t^2}\,dt$ (1)

$v(t) = -\frac{3}{t} + C \Big|_{\substack{t=1 \\ v=2}}$ (1) (1) $\Rightarrow -3 + C \overset{\text{set}}{=} 2 \Rightarrow C = 5$ (1) (1) $\therefore$

$v(t) = \frac{-3}{t} + 5$ (1)

5 (B) $\frac{ds}{dt} = v(t) = \frac{-3}{t} + 5$

$\int ds = \int \frac{-3}{t} + 5\,dt$ (1)

$x(t) = -3\ln t + 5t + D \Big|_{\substack{t=1 \\ x(t)=6}}$ (1) (1) $\Rightarrow 5 + D \overset{\text{set}}{=} 6 \Rightarrow D = 1$ (1)

$\therefore x(t) = -3\ln t + 5t + 1$ (1)

2 (C) $x(e) = -3\ln e + 5e + 1 =$ (1)

$5e - 2.$ (1)

1982
(2)

(2)

$$f(x) = \frac{x^3 - x}{x^3 - 4x}$$

(1)

(1) A) $\lim_{x \to 0} \frac{x^3 - x}{x^3 - 4x} = \lim_{x \to 0} \frac{x(x^2-1)}{x(x^2-4)} = \lim_{x \to 0} \frac{x^2-1}{x^2-4} = \frac{-1}{-4} = \frac{1}{4}$

(3) B) $f(x) \overset{\text{set}}{=} 0 \Rightarrow$ Numerator $\overset{\text{set}}{=} 0 = x(x^2-1)$ (1)

(1) $x = 0$ $\quad x = \pm 1$ $\quad f(x) = 0 \ \forall x \in \{-1, 1\}$ (1)

Reject Not element of Domain.

(6) C) Vertical asymptotes $\Rightarrow$ where denominator $= 0$

$x(x-2)(x+2)$ (1)

reject → (1) explain (1) Not a vertical asymptote Simply a "hole" in the graph

(1) equat (1) values (1)

$x = 2$, $x = -2$ Vertical asymptotes,

$$\begin{array}{r} 1 \\ x^3 - 4x \overline{) x^3 - x} \\ \underline{x^3 - 4x} \\ 3x \end{array} \Rightarrow 1 + \frac{3x}{x^3 - 4x}$$

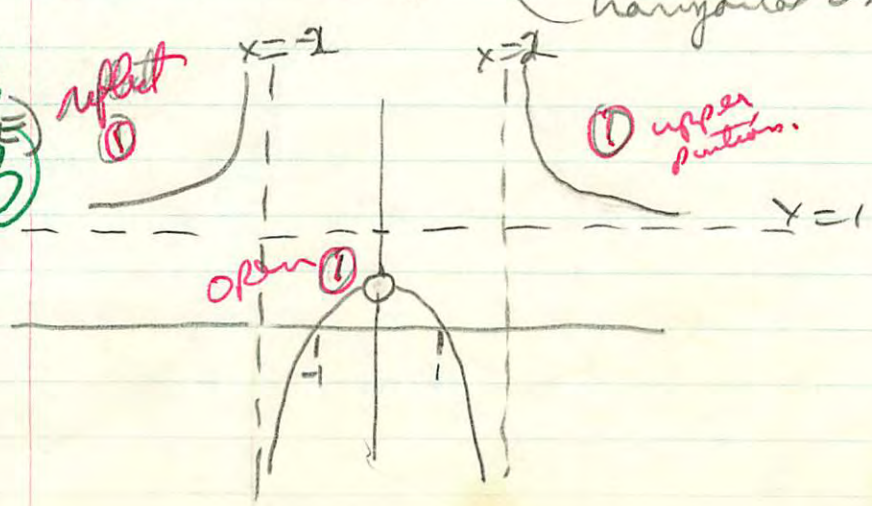
(1) Justify

$\lim_{x \to \pm\infty} 1 + \frac{3x}{x^3 - 4x} = 1$ is horizontal asymptote $y = 1$ (1) equat

D) Since $f(x) = f(-x)$

(2) f is symmetric WRT y-axis,

(3) (3) E) reflect (1)



(1) upper portion.

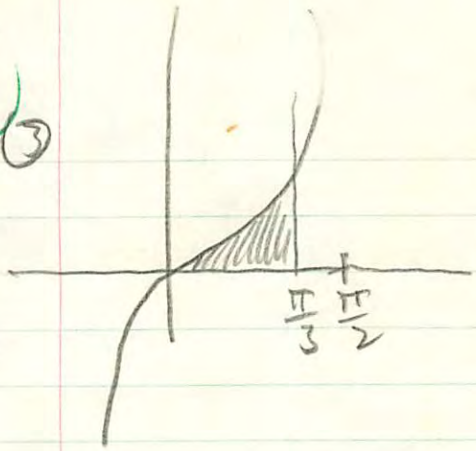
$\frac{x(x^2-1)}{x(x^2-4)}$ $\quad x > 2$ $\quad \frac{+\ +}{+\ +} = +$

$1 < x < 2$ $\quad \frac{+\ +}{+\ -} = -$

1982

(3)

(7) (3)



limits (1)

A) $\int_0^{\frac{\pi}{3}} \tan x \, dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$ (1) (1)

New limits ~~[illegible]~~ (1)

(1) Let $u = \cos x$ $\quad x = 0 \quad u = 1$

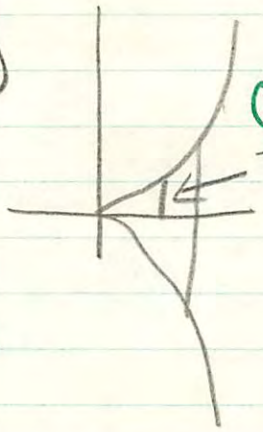
$du = -\sin x \, dx \quad x = \frac{\pi}{3} \quad u = \frac{1}{2}$

$-\int_1^{\frac{1}{2}} \frac{du}{u} = +\int_{\frac{1}{2}}^{1} \frac{du}{u} = \ln u \Big|_{\frac{1}{2}}^{1} =$ (1)

$-\ln \frac{1}{2} = -\ln 2^{-1} = \ln 2,$

(area required is $\ln 2$ (1) answers question.

(8) B)



(1) Typical radius $= \tan x$

(1) $\pi \int r^2 dx \Rightarrow$

fct (1) (1)

$\pi \int_0^{\frac{\pi}{3}} \tan^2 x \, dx = \pi \int_0^{\frac{\pi}{3}} \sec^2 x - 1 \, dx$

limits (1)

$s^2 + c^2 = 1$

$t^2 + 1 = \sec^2$

~~[illegible]~~

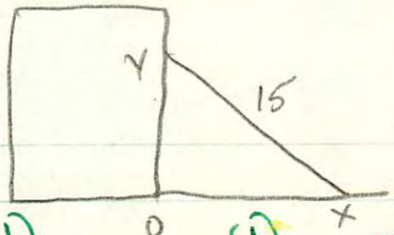
(1) (1)

$\pi \left[\tan x - x \Big|_0^{\frac{\pi}{3}} \right] = \pi \left[\sqrt{3} - \frac{\pi}{3} \right] =$

$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$ Volume of revolution. (1)

1982
(4)

(8) 4.



(1) (1)

A) $\frac{dx}{dt} = \frac{1}{2}$ $\frac{dy}{dt} = ?$

$x^2 + y^2 = 15^2$ (1)

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ (1)

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}\Big|_{x=9,\ y=12}$ (1) $= -\frac{9}{12}\cdot\frac{1}{2} = -\frac{3}{8}$ (1)

OY is falling at $\frac{3}{8}$ FT/sec. (1)

$9^2 + y^2 = 15^2$

(1) $y^2 = 144$

(1) $y = 12$

75
15
225
−81
144

(7) B) $\frac{dx}{dt} = \frac{1}{2}$ $\frac{dy}{dt}\Big|_{x=9} = \frac{-3}{8}$ $\frac{dA}{dt} = ?$ (1)

$A = \frac{1}{2} x y$ (1)

$\frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + \frac{1}{2}y\frac{dx}{dt}$ (2) $\frac{dA}{dt}\Big|_{\frac{dx}{dt}=\frac{1}{2},\ x=9,\ \frac{dy}{dt}=-\frac{3}{8},\ y=12}$ =

(1) $= \frac{1}{2}\cdot 9\cdot\left(-\frac{3}{8}\right) + \frac{1}{2}(12)\left(\frac{1}{2}\right)$

$= \frac{-27}{16} + 3 = \frac{-27+48}{16} = \frac{21}{16}$ (1)

Area of Δ XOY increasing at rate of $\frac{21}{16}$ Sq FT/sec. (1)

1982 (5)

(9) (5)

A) $f(x) = (x^2+1)e^{-x} \quad -4 \le x \le 4$

$f'(x) = 2xe^{-x} - e^{-x}(x^2+1) \overset{\text{set}}{=} 0$ (2)

$e^{-x}(2x - x^2 - 1)$

$-e^{-x}(x^2 - 2x + 1)$

(1) $-e^{-x}(x-1)^2$

f' → — x —

(1) $f' \le 0 \ \forall x \in [-4, 4]$

(1) $f(-4) = 17e^4$

(1) $f(4) = 17e^{-4}$

$(-4, 17e^4]$ is Absolute max because fctn continues to fall from $17e^4$ within the given interval. (3)

u v

$f' = -e^{-x}(x-1)^2$

(6) B)

$f''(x) = e^{-x}(x-1)^2 - 2e^{-x}(x-1)$ (2)

$= e^{-x}(x-1)(x-1-2)$

$= e^{-x}(x-1)(x-3) \overset{\text{set}}{=} 0$ (2)

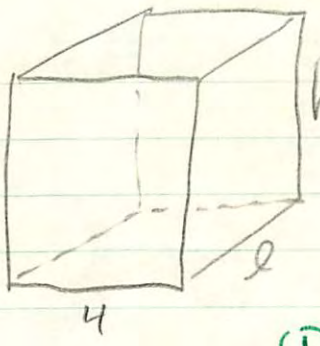
INflection occurs at $x=1$ & $x=3$

(2) if $x \in [1, 3]$

at $x=1$ we have a Lazy PI because $f'(1)$ is also zero.

at $x=3$ we have an actual PI.

(6)



$v = 36$

(1)

$4lh = 36 \Rightarrow lh = 9 \Rightarrow h = \frac{9}{l}$

(1) Base area $4l$
Side area $2lh + 2 \cdot 4h$

(1) $c = (4l)(10) + 5(2lh + 8h)$

(2) $c(l) = 40l + 10l\left(\frac{9}{l}\right) + \frac{40 \cdot 9}{l}$

$= 40l + 90 + \frac{360}{l}$

$c'(l) = 40 - 360l^{-2} \overset{\text{set}}{=} 0$ (1)

$40l^2 = 360$

$l^2 = 9$

(1) $l = \pm 3$

Reject $l = -3$ (1)
since $l > 0$
$\therefore l = 3$ (1)

(3) $c''(l) = 720l^{-3} > 0 \ \forall l > 0 \therefore$ min cost f''test (3)

Dimensions will be $3 \times 3 \times 4$

$C(3) = \text{Cost} = 40 \cdot 3 + 90 + \frac{360}{3}$

$120 + 90 + 120 = \$330$ is the least cost. (1)

(2) Evaluate

Answer the Question.

7B $f(-x) = f(x)$ def of even fctn

$f'(-x) = f'(x)$ der of each side

der of inside $= -1$

def of odd fctn.

$\therefore$ if f is even,
f' is odd.

1982
(7)

(7) $f(-x)=f(x)$ $f(p)=1$ $f'(p)=5$ $p>0$

(5) A) $f'(-p)=?$

up to 5 for Properly Justified correct answer.

f is even fctn $\left(f(x)=f(-x)\right)$ $\therefore$ f' is odd

so $f'(-x) = -f'(x)$ $f'(p)=5$ $\therefore$ $f'(-p)=-5$

$f'(-p) = -f'(p)$

-5

B) $f'(0)=?$

5

Same as A.

f is an even fctn. this means that f cannot have x to the 1st power.

f' is odd however there is no constant added to f'. The lowest possible power of x is 1 in f' thus $f'(0)=0$

$f = \ldots + a_N x^{2N}$

$f' = \ldots + a_N 2N x^{2N-1}$

but $2N-1 \neq 0$ so there is an x in every term.

conclusion $f'(0)$ must $=0$.

5 C) $(-p,1)$ $m=-5$ $(p,1)$ $m=5$

(2) (1) $y-1=-5(x+p)$ (2) $y-1=5(x-p)$

$y=-5x-5p+1$ (1) $y=5x-5p+1$

$-5x-5p+1 \overset{\text{set}}{=} 5x-5p+1$ to find common pt

$-10x=0$

$x=0$

$y=-5\cdot 0-5p+1$ $\therefore$ (1) $y=-5p+1$

intersection point is $(0,-5p+1)$ (1)

FORM 3DBP

1981

The College Board

Advanced Placement Examination

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

Number of problems—7

Per cent of total grade—50

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

Notes: (1) In this examination, ln x denotes the natural logarithm of x (logarithm to the base e). (2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which f(x) is a real number.

1. Let f be the function defined by $f(x) = x^4 - 3x^2 + 2$.

   (a) Find the zeros of f.

   (b) Write an equation of the line tangent to the graph of f at the point where $x = 1$.

   (c) Find the x-coordinate of each point at which the line tangent to the graph of f is parallel to the line $y = -2x + 4$.

---

2. Let R be the region in the first quadrant enclosed by the graphs of $y = 4 - x^2$, $y = 3x$, and the Y-axis.

   (a) Find the area of region R.

   (b) Find the volume of the solid formed by revolving the region R about the X-axis.

---

3. Let f be the function defined by $f(x) = 12x^{\frac{2}{3}} - 4x$.

   (a) Find the intervals on which f is increasing.

   (b) Find the x- and y-coordinates of all relative maximum points.

   (c) Find the x- and y-coordinates of all relative minimum points.

   (d) Find the intervals on which f is concave downward.

   (e) Using the information found in parts (a), (b), (c), and (d), sketch the graph of f on the axes provided.

---

4. Let f be the function defined by $f(x) = 5^{\sqrt{2x^2-1}}$.

   (a) Is f an even or odd function? Justify your answer.

   (b) Find the domain of f.

   (c) Find the range of f.

   (d) Find $f'(x)$.

TURN THE PAGE

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5. Let f be a function defined by

$$f(x) = \begin{cases} 2x + 1, & \text{for } x \leq 2, \\ \frac{1}{2}x^2 + k, & \text{for } x > 2. \end{cases}$$

(a) For what value of k will f be continuous at $x = 2$? Justify your answer.

(b) Using the value of k found in part (a), determine whether f is differentiable at $x = 2$. Use the definition of the derivative to justify your answer.

(c) Let $k = 4$. Determine whether f is differentiable at $x = 2$. Justify your answer.

---

6. A particle moves along the X-axis so that at time t its position is given by $x(t) = \sin(\pi t^2)$ for $-1 \leq t \leq 1$.

(a) Find the velocity at time t.

(b) Find the acceleration at time t.

(c) For what values of t does the particle change direction?

(d) Find all values of t for which the particle is moving to the left.

---

7. Let f be a continuous function that is defined for all real numbers x and that has the following properties.

(i) $\int_1^3 f(x)\,dx = \frac{5}{2}$

(ii) $\int_1^5 f(x)\,dx = 10$

(a) Find the average (mean) value of f over the closed interval $[1, 3]$.

(b) Find the value of $\int_3^5 [2f(x) + 6]dx$.

(c) Given that $f(x) = ax + b$, find the values of a and b.

END OF EXAMINATION

$f(x) = x^4 - 3x^2 + 2$

3 A) $x^4 - 3x^2 + 2 \overset{\text{set}}{=} 0$ (1)

$(x^2-2)(x^2-1) = 0$ (1)

$x = \pm\sqrt{2} \quad x = \pm 1$ (1)

f has zeros at $x = \{\pm\sqrt{2}, \pm 1\}$

5 B) $f(1) = 1 - 3 + 2 = 0 \Rightarrow (1, 0)$

$f'(x) = 4x^3 - 6x \Big| = 4 - 6 = -2$ (2) $x = 1$

$m = -2$ 2

$y - 0 = -2(x-1)$

$y = -2x + 2$, 3

7 C) $y = -2x + 4$

2 $\therefore m = -2$

$f' \Rightarrow 4x^3 - 6x \overset{\text{set}}{=} -2$

$2x^3 - 3x + 1 = 0$

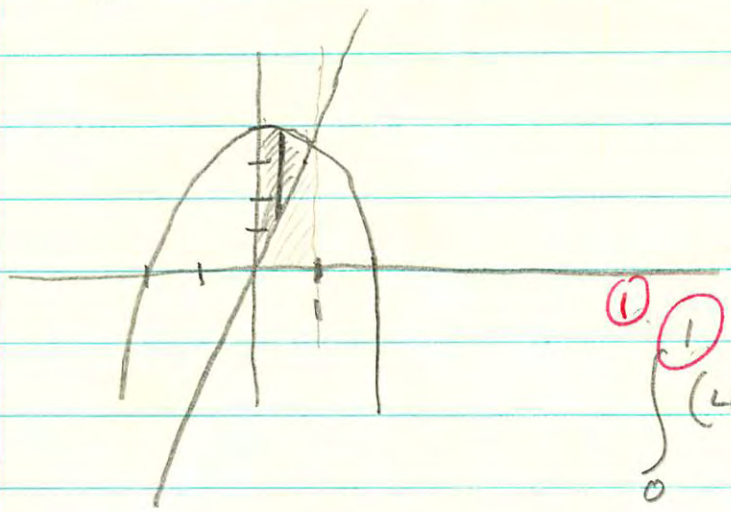
	2	0	−3	1
2	1	2	2	−1
		2	2	0

$2x^2 + 2x - 1$ 3

$$x = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

x values are: $1, \frac{-1 \pm \sqrt{3}}{2}$

$y = 4 - x^2$ $\quad y = 3x$



$3x \overset{\text{Set}}{=} 4 - x^2$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

reject (2) $x = 1$

(1) $\int_0^1 (4 - x^2) - 3x \, dx$ (3)

$\int_0^1 4 - 3x - x^2 \, dx =$

9 B A)

$$4x - \frac{3}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1 = 4 - \frac{3}{2} - \frac{1}{3} = \frac{24 - 9 - 2}{6} = \frac{13}{6}$$

6 B) use washer method.

$$\pi \int_0^1 (4 - x^2)^2 - (3x)^2 \, dx = \pi \int_0^1 16 - 8x^2 + x^4 - 9x^2 \, dx$$

$$\pi \left[16x - \frac{17}{3}x^3 + \frac{1}{5}x^5 \right] \Big|_0^1 = \pi \left(16 - \frac{17}{3} + \frac{1}{5} \right)$$

$\frac{16}{15}$
$\frac{80}{240}$

$$\frac{240 - 85 + 3}{15} = \frac{158\pi}{15}$$

$$\lim_{x \to \infty} 12x^{2/3} - 4x =$$

$$\lim_{x \to \infty} 4x\left(\frac{3}{x^{1/3}} - 1\right)$$

$$\infty(0 - 1) = -\infty$$

$$\lim_{x \to -\infty} 4x\left(\frac{3}{x^{1/3}} - 1\right)$$

$$-\infty(0 - 1) = +\infty$$

1981

(3)

$x\left(\frac{12}{x^{1/3}} - 4\right)$

$* \lim_{x \to -\infty} 12x^{2/3} - 4x$

$4(3x^{2/3} - x)$

$4x(3x^{-1/3} - 1)$

$+ \quad + \quad > +\infty$

$f(x) = 12x^{2/3} - 4x$

A) WTF $\forall x \mid f' > 0$

$f'(x) = 8x^{-1/3} - 4$

$= \frac{8}{x^{1/3}} - 4$ (1)

$\frac{8 - 4x^{1/3}}{x^{1/3}} \overset{set}{=} 0$

$x^{1/3} = 2$

$x = 8$

f': $\pm = -$ | $-$ | 0 | $\pm$ | $+$ | 0 | $-$ | $\mp = -$

$0 \quad 8$

$f' > 0 \quad 0 < x < 8$ (1) (1)

$\therefore$ f increasing for $0 < x < 8$

B) $f' \overset{set}{=} 0 \Rightarrow x = 8$ (f in part A)

$f'' = -\frac{8}{3}x^{-4/3}\Big|_{x=8} < 0 \therefore$

max at $x = 8$ f'' test

$f(8) = 12(8)^{2/3} - 4(8)$

$48 - 32 = 16$

(2) is it pel or absolute? max at $(8, 16)$

(1) Since $f(-1) \overset{Also}{=} 16$ we have only relative max rather than Absolute max.

Notice $f'(0)$ DNE but f decreases to zero $\therefore$ where does it decrease from? $\lim_{x \to -\infty} f(x) \Rightarrow *$

c) only critical value of x yields max. There are no end points. $F'(0)$ does not exist so a possible extremum may occur at $(0, 0)$.

$f(-1) = 12 + 4 = 16$

$f(0) = 0$ (3)

$f(1) = 8 \quad \therefore$ relative min at $(0, 0)$

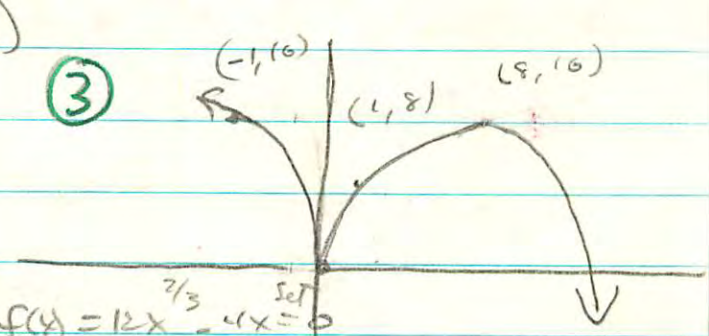
$\lim_{x \to \infty} f(x) \Rightarrow -\infty \therefore$ only relative

D) $f'' = -\frac{8}{3}x^{-4/3} \overset{set}{>} 0$

(3) $f'' < 0 \ \forall x \ (x \neq 0)$

$\therefore$ f is concave down $\forall x \quad x \neq 0$

E) (3)

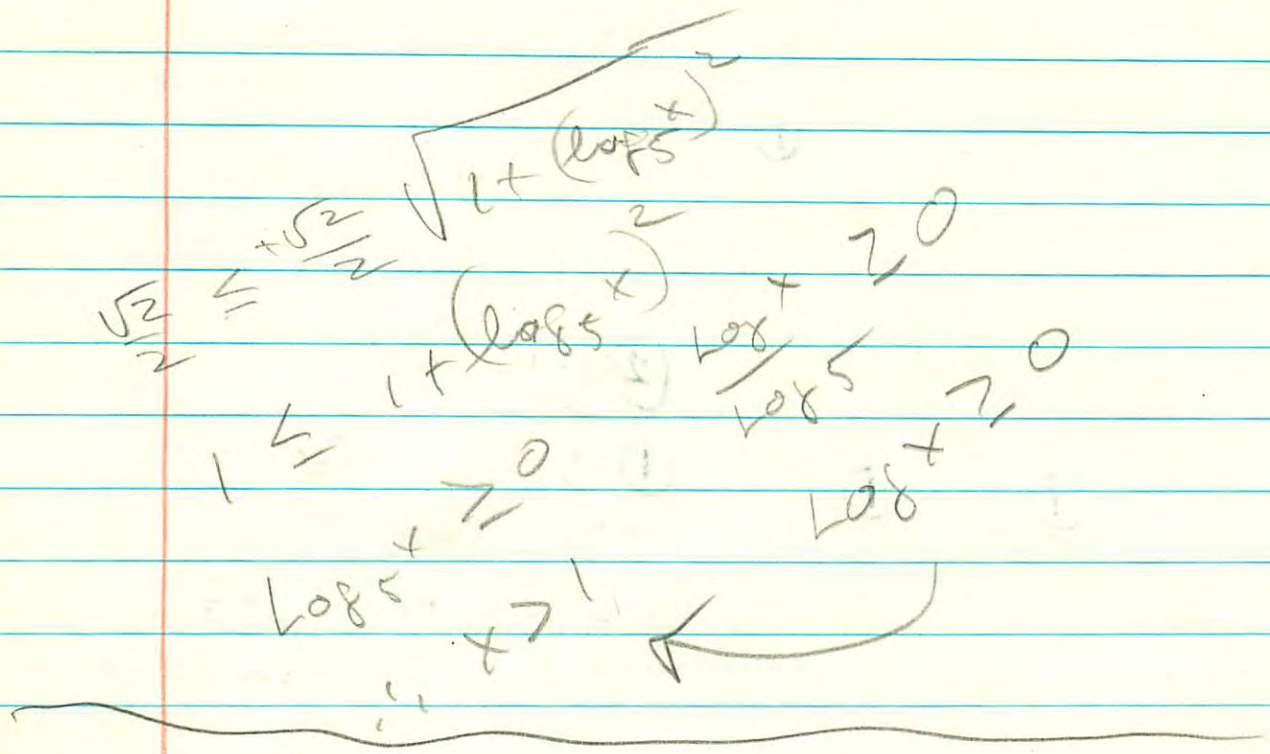


$f(x) = 12x^{2/3} - 4x \overset{set}{=} 0$

$x\left(\frac{3}{x} - 1\right) = 0$

$+ \ (\ -) < 0$

$\lim_{x \to \infty} f(x) < 0$



$\lim_{a \to \infty} 5^{\frac{1}{a}} = 5^0 = 1 \quad \therefore$ Range of $f \geq 1$

#4c Range of $f(x) = 5^{\sqrt{2x^2-1}}$

① $\sqrt{2x^2-1} \geq 0$. ↓ of of sq root.

② let $a = \sqrt{2x^2-1}$

if $a = 0$ $\quad f(x) = 5^0 = 1.$

if $0 < a < 1$ $\quad \lim_{a \to 0} 5^a = 5^0 = 1$ $\quad \lim_{a \to 1} 5^a = 5^1 = 5$ $\Big\} \geq 1$

if $a > 1$ $\quad \lim_{a \to \infty} 5^a \Rightarrow +\infty$ i.e. ≥ 1

Since these are the only possible values of a then the only conclusion is R_f $f(x) \geq 1$

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(4)

(4) $f(x) = 5^{\sqrt{2x^2-1}}$ (2) (1)

A) $f(-x) = 5^{\sqrt{2(-x)^2-1}} = 5^{\sqrt{2x^2-1}} = f(x)$ ∴ f EVEN

(1) By definition "if $f(x) = f(-x)$ then f is Even."

B)

$2x^2 - 1 \geq 0$

$2x^2 \geq 1$

$x^2 \geq \frac{1}{2}$

$x^2 - \frac{1}{2} \geq 0$

$(x - \frac{\sqrt{2}}{2})(x + \frac{\sqrt{2}}{2})$

+ 0 − 0 +

$-\frac{\sqrt{2}}{2}$ 0 $\frac{\sqrt{2}}{2}$

(3) $2x^2 - 1 \geq 0$

$\forall x \mid |x| \geq \frac{\sqrt{2}}{2}$

D_f $x \in \mathbb{R}, |x| \geq \frac{\sqrt{2}}{2}$

C)

Since the exponent ≥ 0

$+^0 = 1$, $+^{\frac{a}{b}} > 0$ $\frac{1}{A}$ $\lim_{a \to \infty} 5$

(3) Range of f is

R_f $f(x) \geq 1$

Also Restrictions on Range of fctn are identical to restrict. on Domain of INVERSE.

Thus,

$\log_5 x$ is defined only for $x > 0$.

∴ range of f NO will be R_f $f(x) > 0$,

D) $f(x) = 5^{\sqrt{2x^2-1}}$

$\ln x = \sqrt{2x^2-1} \ln 5$

$\frac{\ln x}{\ln 5} = \sqrt{2x^2-1}$

$\left(\frac{\ln x}{\ln 5}\right)^2 = 2x^2 - 1$

$2x^2 = \left(1 + \left(\frac{\ln x}{\ln 5}\right)^2\right)$

$x^2 = \frac{1}{2}\left(1 + \left(\frac{\ln x}{\ln 5}\right)^2\right)$

$x = \pm \frac{\sqrt{2}}{2}\sqrt{1 + (\log_5 x)^2}$

$\sin(\log_5 x)$ is squared it becomes > 0. The only other restriction is that x, itself must be > 0. ∴ $R_f \in \mathbb{R}$, $f(x) > 0$

NO.

D) $f'(x)$: (2) $\ln f(x) = \sqrt{2x^2-1} \ln 5$

$\frac{1}{f(x)} f'(x) = \frac{4x \ln 5}{2\sqrt{2x^2-1}}$

$f'(x) = f(x) \frac{2x \ln 5}{\sqrt{2x^2-1}}$ (2)

$f'(x) = \frac{2x\, 5^{\sqrt{2x^2-1}} \ln 5}{\sqrt{2x^2-1}}$ (1)

$$f(x) = \begin{cases} 2x+1 \; ; & x \le 2 \\ \frac{1}{2}x^2 + K \; ; & x > 2 \end{cases}$$

A) (2)

$$\lim_{x\to 2^-} 2x+1 = 5 \overset{\text{set}}{=} \lim_{x\to 2^+} \frac{1}{2}x^2 + K$$

(1) $5 = 2 + K \therefore K = 3.$

(2) If f is continuous at a then $\lim_{x\to a} f(x) \overset{\text{must}}{=} f(a)$.

B) $\lim_{x\to 2} \dfrac{f(x) - f(2)}{x-2}$

(2) I $\lim_{x\to 2^-} \dfrac{2x+1-5}{x-2} = \lim_{x\to 2^-} \dfrac{2x-4}{x-2} = \lim_{x\to 2^-} \dfrac{2(x-2)}{x-2} = 2$

(2) II $\lim_{x\to 2^+} \dfrac{\frac{1}{2}x^2+3-5}{x-2} = \lim_{x\to 2^+} \dfrac{\frac{1}{2}x^2-2}{x-2} = \lim_{x\to 2^+} \dfrac{x^2-4}{2(x-2)} =$

$$\lim_{x\to 2^+} \frac{(x-2)(x+2)}{2(x-2)} = \lim_{x\to 2^+} \frac{x+2}{2} = 2.$$

Since $\lim_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ exists & since this is the definition of the derivative at $x=2$

(2) f is indeed differentiable at $x=2$.

C

if $K=4$ then $f(x) = \begin{cases} 2x+1 \; ; & x \le 2 \\ \frac{1}{2}x^2+4 \; ; & x > 2 \end{cases}$

Since f was continuous when $K=3$ one expects f to be discontinuous at $x=2$ when $K=4$,

(2) the right hand limit: $\lim_{x\to 2^+} \frac{1}{2}x^2+4 = 6$ and left limit is still 5. Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$ f is now not continuous at $x=2$. Since f is not continuous

(2) f is no longer differentiable. A fctn must 1st be Cont. to be differentiable.

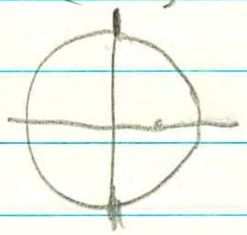
$x(t) = \sin(\pi t^2) \quad -1 \le t \le 1$. (1)

a) Velocity $x'(t) = 2\pi t \cos(\pi t^2)$ (1)

b) Acceleration see below. (3)

c) changes direction when $x'(t)$ changes sign.

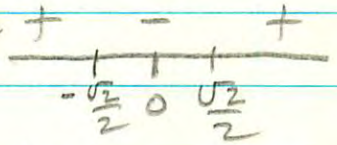
$\cos(\pi t^2) > 0$ for $-\frac{\pi}{2} < \pi t^2 < \frac{\pi}{2}$ $\quad t^2 > 0 \ \forall t$.



$0 < t^2 < \frac{1}{2}$

$t^2 - \frac{1}{2} < 0$

$(t - \frac{\sqrt{2}}{2})(t + \frac{\sqrt{2}}{2}) < 0$



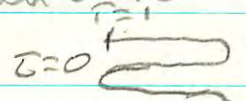
$2\pi t \cos(\pi t^2) = 0$ at $t \in \{-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\}$.

III when $t > 0$ $\cos(\pi t^2) > 0$ for $0 < t < \frac{\sqrt{2}}{2}$ Right

IV when $t > 0$ $\cos(\pi t^2) < 0$ for $\frac{\sqrt{2}}{2} < t \le 1$ Left

I when $t < 0$ $\cos(\pi t^2) < 0$ for $-1 \le t < \frac{\sqrt{2}}{2}$ Right

II when $t \le 0$ $\cos(\pi t^2) > 0$ for $-\frac{\sqrt{2}}{2} < t < 0$ Left



Particle stands to move at $t = -1$.

Particle changes directions at (2)

$t = -\frac{\sqrt{2}}{2}$, $t = 0$, and $t = \frac{\sqrt{2}}{2}$ (2)

Particle stops when $t = 1$.

d) from part c II, IV (2)

Particle is moving Left for $-\frac{\sqrt{2}}{2} < t < 0$ and

$\frac{\sqrt{2}}{2} < t \le 1$ (3)

B) $x'(t) = 2\pi t \cos(\pi t^2)$

(3) $x''(t) = 2\pi \cos(\pi t^2) - 4\pi^2 t^2 \sin(\pi t^2)$

i) $\int_1^3 f(x)\,dx = \frac{5}{2}$

ii) $\int_1^5 f(x)\,dx = 10$

4) $\frac{1}{b-a}\int_a^b f(x)\,dx \Leftarrow$ mean value of f on I.

$\frac{1}{3-1}\int_1^3 f(x)\,dx = \frac{1}{2}\cdot\frac{5}{2} = \frac{5}{4}$ (3) (1)

B) $\int_3^5 [2f(x)+6]\,dx = 2\int_3^5 f(x)\,dx + \int_3^5 6\,dx$ (1)

$2\left[\int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx\right] + \int_3^5 6\,dx$ (2)

$2\left[10 - \frac{5}{2}\right] + 6x\Big|_3^5 = 20-5+30-18 = 27$ (2) (1)

$5-0$
$-2-3$

c) $f(x) = ax+b$

$\int_1^{3,5} ax+b\,dx = \frac{a}{2}x^2+bx\Big|_1^3 = \frac{9a}{2}+3b-\frac{a}{2}-b \overset{\text{set}}{=} \frac{5}{2}$ (1)

$4a+2b = \frac{5}{2}$

$\frac{a}{2}x^2+bx\Big|_1^5 = \frac{25a}{2}+5b-\frac{a}{2}-b \overset{\text{set}}{=} 10$ (1)

$12a+4b = 10$

$-\ 8a-4b = -5$ (2)

$4a = 5$

$a = \frac{5}{4}$

$4\cdot\frac{5}{4}+2b = \frac{5}{2}$

(1) $2b = -\frac{5}{2}$

$b = -\frac{5}{4}$

$a = \frac{5}{4}, \quad b = -\frac{5}{4}$

CK:

$f(x) = \frac{5}{4}x - \frac{5}{4}$ $\quad \int \frac{5}{4}x - \frac{5}{4} = \frac{5}{4}\int x-1\,dx$

$\frac{5}{4}\left(\frac{x^2-2x}{2}\right)\Big|_1^3 = \frac{5}{4}\cdot\left[\frac{3}{2}+\frac{1}{2}\right] = \frac{5}{4}\cdot 2 = \frac{5}{2}$, $\frac{5}{4}\left(\frac{x^2-2x}{2}\right)\Big|_1^5 = \frac{5}{4}\left(\frac{15}{2}+\frac{1}{2}\right) = \frac{5}{4}\cdot 8 = 10$

FORM 3CBP

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Advanced Placement Examination

CALCULUS AB

SECTION II

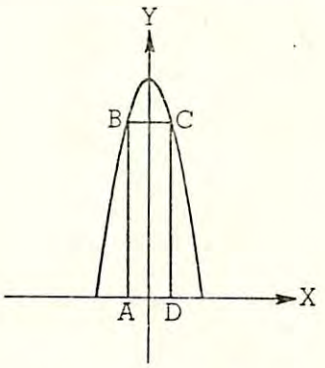
Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Let R be the region enclosed by the graphs of $y = x^3$ and $y = \sqrt{x}$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated by revolving R about the X-axis.



2. A rectangle ABCD with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the X-axis as shown in the figure above.

   (a) Find the x- and y-coordinates of C so that the area of rectangle ABCD is a maximum.

   (b) The point C moves along the curve with its x-coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle ABCD when $x = \frac{1}{2}$.

---

3. Let $f(x) = \ln(x^2)$ for $x > 0$ and $g(x) = e^{2x}$ for $x \geq 0$. Let H be the composition of f with g, that is $H(x) = f(g(x))$, and let K be the composition of g with f, that is $K(x) = g(f(x))$.

   (a) Find the domain of H and write an expression for $H(x)$ that does not contain the exponential function.

   (b) Find the domain of K and write an expression for $K(x)$ that does not contain the exponential function.

   (c) Find an expression for $f^{-1}(x)$, where f^{-1} denotes the inverse function of f, and find the domain of f^{-1}.

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AB 1980

4. The acceleration of a particle moving along a straight line is given by $a = 10e^{2t}$.

   (a) Write an expression for the velocity v, in terms of time t, if $v = 5$ when $t = 0$.

   (b) During the time when the velocity increases from 5 to 15, how far does the particle travel?

   (c) Write an expression for the position s, in terms of time t, of the particle if $s = 0$ when $t = 0$.

---

5. Given the function f defined by $f(x) = \cos x - \cos^2 x$ for $-\pi \leq x \leq \pi$.

   (a) Find the x-intercepts of the graph of f.

   (b) Find the x- and y-coordinates of all relative maximum points of f. Justify your answer.

   (c) Find the intervals on which the graph of f is increasing.

   (d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.

---

6. Let $y = f(x)$ be the continuous function that satisfies the equation $x^4 - 5x^2y^2 + 4y^4 = 0$ and whose graph contains the points (2, 1) and (−2, −2). Let ℓ be the line tangent to the graph of f at $x = 2$.

   (a) Find an expression for y'.

   (b) Write an equation for line ℓ.

   (c) Give the coordinates of a point that is on the graph of f but is not on line ℓ.

   (d) Give the coordinates of a point that is on line ℓ but is not on the graph of f.

---

7. Let p and q be real numbers and let f be the function defined by $f(x) = \begin{cases} 1 + 2p(x-1) + (x-1)^2, & \text{for } x \leq 1 \\ qx + p, & \text{for } x > 1. \end{cases}$

   (a) Find the value of q, in terms of p, for which f is continuous at $x = 1$.

   (b) Find the values of p and q for which f is differentiable at $x = 1$.

   (c) If p and q have the values determined in part (b), is f'' a continuous function? Justify your answer.

END OF EXAMINATION

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FORM 3CBP

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Advanced Placement Examination

CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. The acceleration of a particle moving along a straight line is given by $a = 10e^{2t}$.

   (a) Write an expression for the velocity v, in terms of time t, if $v = 5$ when $t = 0$.

   (b) During the time that the velocity increases from 5 to 15, how far does the particle travel?

   (c) Write an expression for the position s, in terms of time t, of the particle if $s = 0$ when $t = 0$.

---

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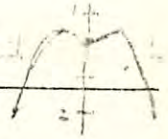
   (a) Find the x-intercepts of the graph of f.

   (b) Find the x- and y-coordinates of all relative maximum points of f. Justify your answer.

   (c) Find the intervals on which the graph of f is increasing.

   (d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.



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3. (a) Determine whether the series $A = \sum_{n=1}^{\infty} \frac{4n}{n^2+1}$ converges or diverges. Justify your answer.

   (b) If S is the series formed by multiplying the nth term in A by the nth term in $\sum_{n=1}^{\infty} \frac{1}{2n}$, write an expression using summation notation for S.

   (c) Determine whether the series S found in part (b) converges or diverges. Justify your answer.

---

4. Let $y = f(x)$ be the continuous function that satisfies the equation $x^4 - 5x^2y^2 + 4y^4 = 0$ and whose graph contains the points (2, 1) and (−2, −2). Let ℓ be the line tangent to the graph of f at $x = 2$.

   (a) Find an expression for y'.

   (b) Write an equation for line ℓ.

   (c) Give the coordinates of a point that is on the graph of f but is not on line ℓ.

   (d) Give the coordinates of a point that is on line ℓ but is not on the graph of f.

This insert may be used for reference and/or scratchwork as you answer the free-response questions, but be sure to show all your work on problems and write your answers in the pink booklet. No credit will be given for work shown on this green insert.

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BC
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(a) Find the general solution of the differential equation $xy' + y = 0$.

(b) Find the general solution of the differential equation $xy' + y = 2x^2y$.

(c) Find the particular solution for the differential equation in part (b) that satisfies the condition that $y = e^2$ when $x = 1$.

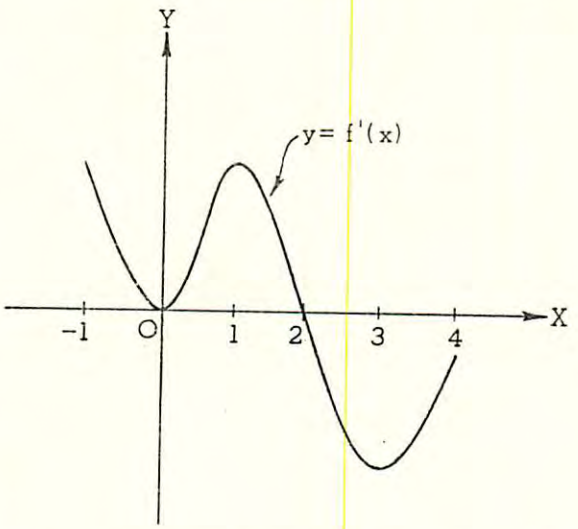
---

6. Let R be the region enclosed by the graphs of $y = e^{-x}$, $x = k$ $(k > 0)$, and the coordinate axes.

(a) Write an improper integral that represents the limit of the area of the region R as k increases without bound and find the value of the integral if it exists.

(b) Find the volume, in terms of k, of the solid generated if R is rotated about the Y-axis.

(c) Find the volume, in terms of k, of the solid whose base is R and whose cross sections perpendicular to the X-axis are squares.



Note: This is the graph of the derivative of f, NOT the graph of f.

7. Let f be a function that has domain the closed interval $[-1, 4]$ and range the closed interval $[-1, 2]$. Let $f(-1) = -1$, $f(0) = 0$, and $f(4) = 1$. Also let f have the derivative function f' that is continuous and that has the graph shown in the figure above.

(a) Find all values of x for which f assumes a relative maximum. Justify your answer.

(b) Find all values of x for which f assumes its absolute minimum. Justify your answer.

(c) Find the intervals on which f is concave downward.

(d) Give all values of x for which f has a point of inflection.

(e) On the axes provided, sketch the graph of f.

END OF EXAMINATION

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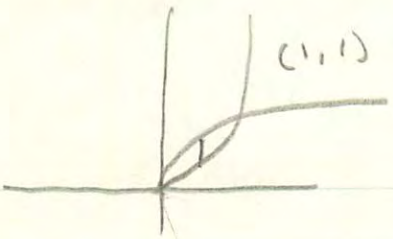
①

① $y = x^3$ $y = \sqrt{x}$



6 a) $A(A) = \int_0^1 \sqrt{x} - x^3 \, dx = \frac{2x^{3/2}}{3} - \frac{x^4}{4}\Big|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$

4 2

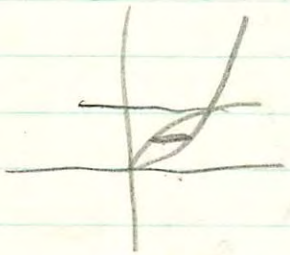
9 b) Volume. revolve $y = \sqrt{x}$ and find volume then revolve $y = x^3$ and find volume
Required volume is the difference.
Vertical washers summed fm $x=0$ to $x=1$

$$\pi \int_0^1 (r_1)^2 - (r_2)^2 \, dx \Rightarrow \pi \int_0^1 (x - x^6) \, dx =$$

$$\pi \left(\frac{x^2}{2} - \frac{x^7}{7} \right)\Big|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{7} \right) = \frac{5\pi}{14}$$ 3

6

$$2\pi \int_0^1 y \left(y^{\frac{1}{3}} - y^2 \right) dy.$$



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$$2\pi \int_0^1 y^{4/3} - y^3 \, dy = 2\pi \left(\frac{3y^{7/3}}{7} - \frac{y^4}{4} \right)\Big|_0^1$$

$$2\pi \left(\frac{3}{7} - \frac{1}{4} \right)$$

$$2\pi \left(\frac{12-7}{28} \right) = \frac{5\pi}{14}$$

1980
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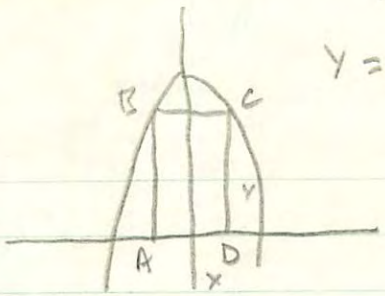
(2)



$y = -4x^2 + 4$

6 a) Area $= 2xy$

$= 2x(-4x^2+4)$

(1) $A(x) = 8x - 8x^3$

$A'(x) = 8 - 24x^2$ set

(2) $8(1-3x^2) = 0$

$x = \pm \frac{\sqrt{3}}{3}$

$A'' = -48x \Big|_{x=\frac{\sqrt{3}}{3}} < 0$ (1)

$\therefore$ area will be a max by f'' test

$y = -4\left(\frac{1}{3}\right) + 4$

$= -\frac{4}{3} + 4$

$= \frac{8}{3}$

"C" $\Rightarrow \left(\frac{\sqrt{3}}{3}, \frac{8}{3}\right)$ (1) (1)

9 B) $\frac{dx}{dt} = 2$ 2pts find $\frac{dA}{dt}\Big|_{x=\frac{1}{2}}$

$A(x) = 8x - 8x^3$

(5) $\frac{dA}{dt} = \left(8\frac{dx}{dt} - 24x^2\frac{dx}{dt}\right)_{x=\frac{1}{2}}$

$\frac{dx}{dt} = 2$

$16 - 24 \cdot \frac{1}{4} \cdot 2 = 16 - 12 = 4$

$\frac{dA}{dt} = 4$ Sq units per second 2pts.

at this instant Rate of change of area is 4 Sq units per second.

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3

$f(x) = \ln(x^2) \quad \forall x > 0$

$g(x) = e^{2x} \quad \forall x \geq 0$

$h(x) = f(g(x))$

$K(x) = g(f(x))$

$D_f \quad x > 0 \qquad R_f \; f(x) \in \mathbb{R}$

$D_g \quad x \geq 0 \qquad R_g \; g(x) \geq 1$

8

a) $D_h \quad x \in \mathbb{R}$ (2) $x \geq 0$ Note: The Range of g Does not violate D_f ∴ No Additional restrictions

$h(x) = \ln(e^{2x})^2 = \ln e^{4x} = 4x \ln e = 4x$ (1) (1) (1)

since $\ln e = 1$ by definition

b) $D_K \quad x > 0 \cap$ $f(x) \geq 0$

$\ln(x^2) \geq 0$!

$2\ln x \geq 0 \Rightarrow x \geq 1$ (2)

∴ $D_K \; x \geq 1$

$K(x) = e^{2(\ln x^2)} = e^{\ln x^4} = x^4$ (1) (1)

7 c) $f(x) = \ln x^2$

$x = \ln z^2$ ½

$e^x = z^2$

$z = e^{\frac{x}{2}} = f^{-1}(x)$ (3)

Domain of inverse has the same restrictions as the range of the original ∴

(2)

$D_{f^{-1}(x)} \quad x \in \mathbb{R},$

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(4)

$a = 10e^{2t} = \frac{dv}{dt}$

(a) $\int dv = \int 10e^{2t}\,dt \Rightarrow v = 5e^{2t} + c$ (3) (1)

$v = 5,\ t = 0 \therefore c = 0 \quad v = 5e^{2t}$ (1)

(b) $v = 5e^{2t}$

$5 = 5e^{2t} \Rightarrow t = 0$ (1)

$15 = 5e^{2t}$ (1)

(3) $3 = e^{2t}$

$\ln 3 = 2t \Rightarrow t = \ln\sqrt{3}$.

s = dist $\frac{ds}{dt} = v$ (1) $\therefore \int v\,dt$ (1) $= s$

$\int ds = \int_0^{\ln\sqrt{3}} 5e^{2t}\,dt \Rightarrow s = \frac{5}{2}e^{2t}\Big|_0^{\frac{1}{2}\ln 3}$ (1)

$\frac{5}{2}\left(e^{\ln 3} - 1\right) = \frac{5}{2}(3-1) = 5$ (1) (1)

particle moves 5 units.

(2) (c) $s = \frac{5}{2}e^{2t} + c$ (from part b)

$s = 0,\ t = 0 \Rightarrow c = -\frac{5}{2} \therefore$

$s = \frac{5}{2}\left(e^{2t} - 1\right)$

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#13
5

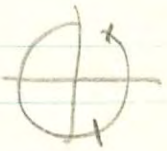
$f(x) = \cos x - \cos^2 x \quad -\pi \le x \le \pi$

(2) a) Solve $\cos x - \cos^2 x \overset{\text{set}}{=} 0$

$\cos x (1 - \cos x) = 0$

$-\frac{\pi}{2}, \frac{\pi}{2}, 0 \qquad \left(-\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), (0,0)$

6 b) $f'(x) = -\sin x + 2\cos x \sin x \overset{\text{set}}{=} 0$



(2) $\sin x (2\cos x - 1) = 0$

$\sin x = 0 \quad x \in \{-\pi, 0, \pi\}$ $\qquad 2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$x \in \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\}$

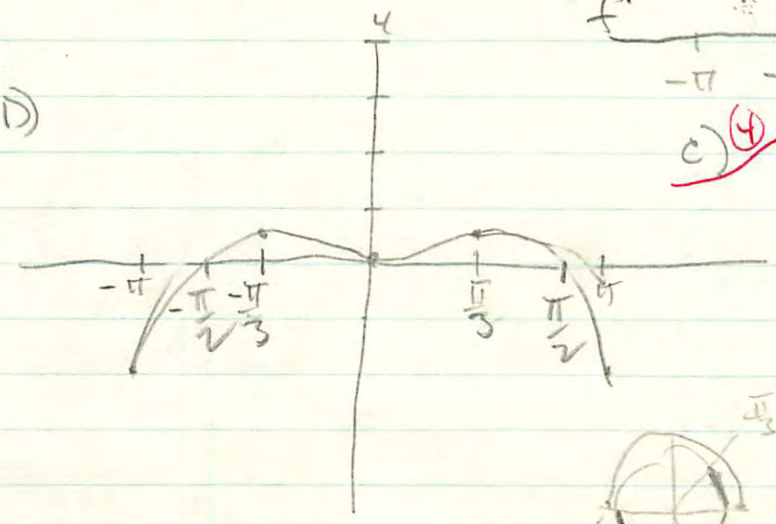
Justification (2)

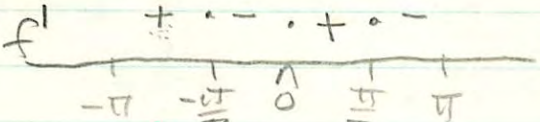
$f(-\pi) = -1 - (-1)^2 = -2$	f test.
$f(-\frac{\pi}{3}) = \frac{1}{2} - (\frac{1}{4}) = \frac{1}{4}$	
$f(0) = 1 - 1 = 0$	
$f(\frac{\pi}{3}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$	
$f(\pi) = -1 - (-1)^2 = -2$	

using f' test we increase to $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$ from the left and right.

we do have $(-\frac{\pi}{3}, \frac{1}{4})$ and $(+\frac{\pi}{3}, \frac{1}{4})$ as relative max for the fctn (2)

(3) D)





c) (4) $\sin x (2\cos x - 1) > 0$ (2)

$\sin x > 0 \quad 2\cos x - 1 > 0$

$\cos x > \frac{1}{2}$

0 →

fctn is increasing for $x \in (-\pi, -\frac{\pi}{3})$ (1) and $(0, \frac{\pi}{3})$ (1)

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6

$x^4 - 5x^2y^2 + 4y^4 = 0$ (2)

$4x^3 - 10x^2y\frac{dy}{dx} - 10xy^2 + 16y^3\frac{dy}{dx} = 0$

(3) a) $\frac{dy}{dx} = \frac{10xy^2 - 4x^3}{16y^3 - 10x^2y} = \frac{5xy^2 - 2x^3}{8y^3 - 5x^2y}$ (1)

b) $x = 2 \Rightarrow 16 - 20y^2 + 4y^2 = 0$ (3)

$4(y^4 - 5y^2 + 4) = 0$

$y(y^2-1)(y^2-4) = 0 \Rightarrow y \in \{\pm 1, \pm 2\}$

$x^4 - 5x^2y^2 + 4y^4 = 0$

$(x^2 - y^2)(x^2 - 4y^2) = (x-y)(x+y)(x-2y)(x+2y) = 0$ (2)

4 intersecting lines.

$\left.\frac{dy}{dx}\right|_{2,1} = \frac{5 \cdot 2 - 16}{8 - 20}$ $\frac{-6}{-12} = \frac{1}{2}$

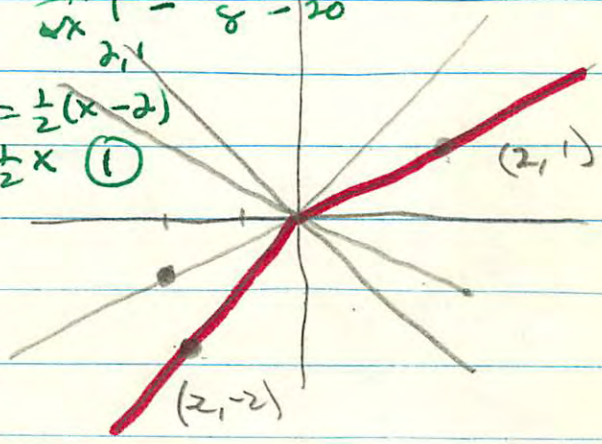
we are given coord (2, 1)

at $x = 2$ tangent is the line of the graph $y = \frac{1}{2}x$ for $x \geq 0$

$y - 1 = \frac{1}{2}(x - 2)$

$y = \frac{1}{2}x$ (1)

(4) Thus tangent has equation $y = \frac{1}{2}x$



3 c) $(-2, -2)$ is on f

(6) (3) but not on l Show lines

This shows recognition that graph has change.

2 (3) d) $(-2, -1)$ is on l

(2) but not on f.

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(7)

$$f(x) = \begin{cases} 1 + 2p(x-1) + (x-1)^2 & \text{if } x \le 1 \\ qx + p & \text{if } x > 1 \end{cases}$$

(1) $f(1) = 1$

6a) f is cont at $x = 1$

$$\lim_{x \to 1^-} 1 + 2p(x-1) + (x-1)^2 = \lim_{x \to 1^+} qx + p = 1$$

$\Leftarrow$ (3) $\rightarrow$

$$1 = q + p = 1$$

(2)

$\therefore q = 1 - p$

6 b)

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{1 - 2p(x-1) + (x-1)^2 - 1}{x - 1} =$$

$$\lim_{x \to 1^-} +2p + (x-1) = +2p \quad (2)$$

$$\lim_{x \to 1^+} \frac{qx + p - q - p}{x - 1} = \lim_{x \to 1} \frac{qx - q}{x - 1} = \lim_{x \to 1} q = q \quad (2)$$

(1)

take diff $2p \overset{\text{must}}{=} q$ but $q = 1 - p$ so

$$2p = 1 - p$$

$$3p = 1 \quad p = \tfrac{1}{3} \quad (1) \quad q = \tfrac{2}{3}$$

3 (c)

$$f(x) = \begin{cases} 1 + \frac{2}{3}(x-1) + (x-1)^2 & x \le 1 \\ \frac{2}{3}x + \frac{1}{3} & x > 1 \end{cases}$$

$$f'(x) = \begin{cases} \frac{2}{3} + 2(x-1) & x \le 1 \\ \frac{2}{3} & x > 1 \end{cases}$$

(2)

$$f'' = \begin{cases} 2 & x \le 1 \\ 0 & x > 1 \end{cases}$$

(1) f'' is not a continuous fctn since we have two constant fctns which are obviously not continuous.

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Advanced Placement Examination

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Given the function f defined by $f(x) = 2x^3 - 3x^2 - 12x + 20$.
   (a) Find the zeros of f.
   (b) Write an equation of the line normal to the graph of f at $x = 0$.
   (c) Find the x- and y-coordinates of all points on the graph of f where the line tangent to the graph is parallel to the X-axis.

---

2. A function f is defined by $f(x) = xe^{-2x}$ with domain $0 \leq x \leq 10$.
   (a) Find all values of x for which the graph of f is increasing and all values of x for which the graph is decreasing.
   (b) Give the x- and y-coordinates of all absolute maximum and minimum points on the graph of f. Justify your answers.

---

3. Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.

---

4. A particle moves along a line so that at any time t its position is given by $x(t) = 2\pi t + \cos 2\pi t$.
   (a) Find the velocity at time t.
   (b) Find the acceleration at time t.
   (c) What are all values of t, $0 \leq t \leq 3$, for which the particle is at rest?
   (d) What is the maximum velocity?

---

5. Let R be the region bounded by the graph of $y = \frac{1}{x} \ln x$, the X-axis, and the line $x = e$.
   (a) Find the area of the region R.
   (b) Find the volume of the solid formed by revolving the region R about the Y-axis.

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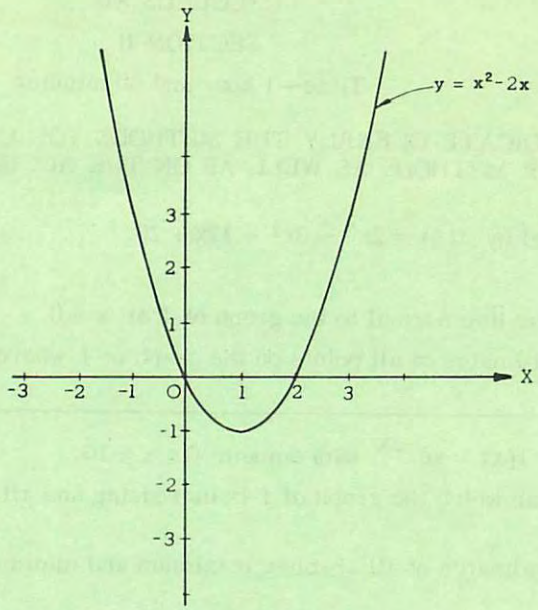

6. The curve in the figure represents the graph of f, where $f(x) = x^2 - 2x$ for all real numbers x.

   (a) On the axes provided, sketch the graph of $y = |f(x)|$.

   (b) Determine whether the derivative of $|f(x)|$ exists at $x = 0$. Justify your answer.

   (c) On the axes provided, sketch the graph of $y = f(|x|)$.

   (d) Determine whether $f(|x|)$ is continuous at $x = 0$. Justify your answer.

---

7. Let f be the function defined by $f(x) = x^3 + ax^2 + bx + c$ and having the following properties.

   (i) The graph of f has a point of inflection at $(0, -2)$.

   (ii) The average (mean) value of $f(x)$ on the closed interval $[0, 2]$ is -3.

   (a) Determine the values of a, b, and c.

   (b) Determine the value of x that satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[0, 3]$.

END OF EXAMINATION

The College Board

Advanced Placement Examination

CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. A particle moves along a line so that at any time t its position is given by $x(t) = 2\pi t + \cos 2\pi t$.
   (a) Find the velocity at time t.
   (b) Find the acceleration at time t.
   (c) What are all values of t, $0 \leq t \leq 3$, for which the particle is at rest?
   (d) What is the maximum velocity?

---

2. Given the differential equation $py'' + y' - 2y = qx$.
   (a) Find the general solution of the differential equation when $p = 0$ and $q = 0$.
   (b) Find the general solution of the differential equation when $p = 1$ and $q = 0$.
   (c) Find the general solution of the differential equation when $p = 1$ and $q = 2$.

---

3. Find the maximum volume of a box that can be made by cutting out squares from the corners of an 8-inch by 15-inch rectangular sheet of cardboard and folding up the sides. Justify your answer.

---

4. Let f be the function defined by $f(x) = \frac{1}{1 - 2x}$.
   (a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 0$.
   (b) What is the interval of convergence for the series found in part (a)? Show your method.
   (c) Find the value of f at $x = -\frac{1}{4}$. How many terms of the series are adequate for approximating $f\left(-\frac{1}{4}\right)$ with an error not exceeding one per cent? Justify your answer.

---

5. Let R be the region bounded by the graph of $y = \frac{1}{x} \ln x$, the X-axis, and the line $x = e$.
   (a) Find the area of the region R.
   (b) Find the volume of the solid formed by revolving the region R about the Y-axis.

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6. A particle moves in the XY-plane so that at any time $t \geq 0$ its position (x, y) is given by $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$.

(a) Find the velocity vector for any $t \geq 0$.

(b) Find $\lim_{t \to +\infty} \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

(c) The particle moves on a hyperbola. Find an equation for this hyperbola in terms of x and y.

(d) On the axes provided, sketch the path of the particle showing the velocity vector for $t = 0$.

---

7. Let f be a function with domain the set of all real numbers and having the following properties.

(i) $f(x + y) = f(x)f(y)$ for all real numbers x and y.

(ii) $\lim_{h \to 0} \frac{f(h) - 1}{h} = k$, where k is a nonzero real number.

(a) Use these properties and a definition of the derivative to show that $f'(x)$ exists for all real numbers x.

(b) Let $f^{(n)}$ denote the nth derivative of f. Write an expression for $f^{(n)}(x)$ in terms of $f(x)$.

(c) Given that $f(1) = 2$, use the Mean Value Theorem to show that there exists a number c such that $0 < c < 3$ and $f'(c) = \frac{7}{3}$.

END OF EXAMINATION

1979
AB
①

$f(x) = 2x^3 - 3x^2 - 12x + 20$ $\quad \pm\left\{\frac{1,2,4,5,10,20}{1,2}\right\}$

a)

$\pm\frac{1}{2}, \pm 2, -4, -\frac{5}{2}, -10,$

$\{+2, -\frac{5}{2}\}$ multiplicity of 2

	2	−3	−12	20
−1	2	−5	−7	27
1	2	−1	−13	7
~~2~~	2	1	−10	0
$-\frac{5}{2}$	2	−8	8	0
	1	−4	4	

$x^2 - 4x + 4$

b) find equat ⊥ to tangt at $x=0$

$f'(x) = 6x^2 - 6x - 12$ $\quad 2\ -3\ -12\ 20$ $\quad (x-2)^2$

$= 6(x^2 - x - 2)$ $\quad 0\ \ 0\ \ -12\ \ 20$

$6(x-2)(x+1)$

$f'(0) = 6(-2)(1) = -12 \therefore \perp m = +\frac{1}{12}$

Solution Set $x \in \{2, -\frac{5}{2}\}$

Note: 2 has a multiplicity of 2. (remainder not 0)

$f(0) = 20$

$y - 20 = \frac{1}{12}x \Rightarrow y = \frac{1}{12}x + 20$

c) $f'(x) \overset{\text{Set}}{=} 0 \Rightarrow x \in \{2, -1\}$

$(-1, 27)$

$(2, 0)$ This is the x axis

1979
AB
(2)

$f(x) = \overset{u}{x}\overset{v}{e^{-2x}}$ $\quad 0 \le x \le 10$

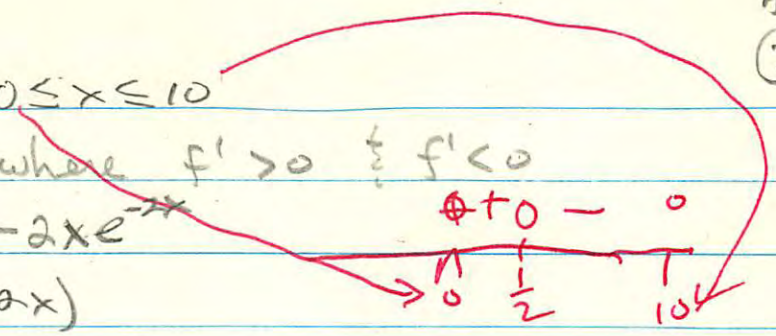
(a) find intervals where $f' > 0$ & $f' < 0$

$f'(x) = e^{-2x} \quad -2xe^{-2x}$

$e^{-2x}(1-2x)$

$+ \quad + \text{ if } x < \frac{1}{2}$ f increasing $0 \le x < \frac{1}{2}$

$- \text{ if } x > \frac{1}{2}$ f decreasing $\frac{1}{2} < x \le 10$



b) critical value of $x \Rightarrow \frac{1}{2}$

using f test

$f(0) = 0$ abs min $(0, 0)$

$f(\frac{1}{2}) = \frac{1}{2}e^{-1}$ abs max $(\frac{1}{2}, \frac{1}{2e})$

$f(10) = 10e^{-5}$

using f' test, + then − at $x = \frac{1}{2}$ ∴ ✗ max

end pt. ⟶ increase fm 0 ∴ min at $x = 0$

⟶ decrease to $\frac{10}{e^{20}}$ ∴ min at $x = 10$

$\frac{1}{2}, \frac{1}{2e}$ Ab max – only max.

$(0,0)$ $(10, \frac{10}{e^{20}})$ since $\frac{10}{e^{20}} > 0$ this is only a relative min

So required absolute min is $(0,0)$.

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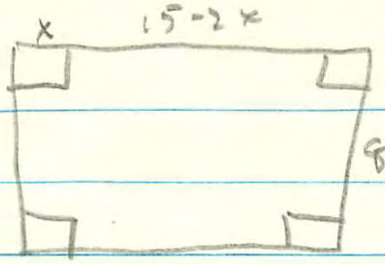
AB
(3)

$2 \times 11 \times 26$
$1 \times 13 \times 6$ (C)



$V = x(15-2x)(8-2x)$

$V = 4x^3 - 46x^2 + 120x$

$V' = 12x^2 - 92x + 120 \overset{\text{set}}{=} 0$

$3x^2 - 23x + 30 = 0$

$(3x-5)(x-6) = 0$

$x = \frac{5}{3} \quad x = 6$ reject $x < 4$

$V'' = 24x - 92$

$V''\left(\frac{5}{3}\right) = 40 - 92 < 0$ max

$V\left(\frac{5}{3}\right) = \frac{5}{3}\left(15 - \frac{10}{3}\right)\left(8 - \frac{10}{3}\right)$

$= \frac{5}{3}\left(\frac{35}{3}\right)\left(\frac{14}{3}\right)$

$\frac{2450}{27} \approx 90.74$

1979
AB
(4)

$x(t) = 2\pi t + \cos 2\pi t$

a) Velocity $= x'(t)$

$x'(t) = 2\pi - 2\pi \sin 2\pi t = 2\pi(1 - \sin 2\pi t)$

b) acc $= x''(t)$

$x''(t) = -4\pi^2 \cos 2\pi t$

c) Velocity $= 0$ $\quad x'(t) \overset{\text{set}}{=} 0 \quad 2\pi(1 - \sin(2\pi t)) \overset{\text{set}}{=} 0$

$t \in [0, 3]$ $\quad \sin(2\pi t) = 1$ $\quad 2\pi t \in \left\{\frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}\right\}$

$t = \frac{k}{2}$

Particle at rest when

$t \in \left\{\frac{1}{4}, \frac{5}{4}, \frac{9}{4}\right\}$

$t \in \left\{\frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \frac{13}{4}\right\}$

reject outside of interval

d) max velocity?

Velocity $= 2\pi - 2\pi \sin 2\pi t$

is when $\sin 2\pi t$ is at min value (which is -1)

$\therefore$ max velocity is $2\pi - (-2\pi) = 4\pi$

when $\sin = +1$

$2\pi t \overset{\text{set}}{=} \frac{(2k+1)\pi}{2}$

$4t = 2k+1$

$t = \frac{2k+1}{4}$

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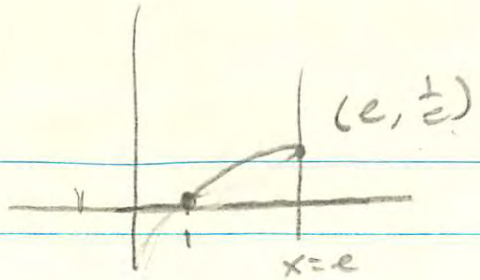
AB
(5)

$y = \frac{1}{x}\ln x$



A) $\int_1^e \frac{1}{x}\ln x\,dx$

let $u = \ln x$ $\quad x=1\ u=0$

$du = \frac{1}{x}dx$ $\quad x=e\ u=1$

$$\int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

(b) Shell

$2\pi\int_1^e$ (radius of Untitled Sample)(height) dx

let $u = \ln x$ $\quad v = x$

$du = \frac{1}{x}dx$ $\quad dv = dx$

$$2\pi\int_1^e x\left(\frac{1}{x}\ln x\right)dx = 2\pi\int_1^e \ln x\,dx$$

$$2\pi\left[x\ln x - x\right]\Big|_1^e = 2\pi\left[e - e - 0 + 1\right] = 2\pi$$

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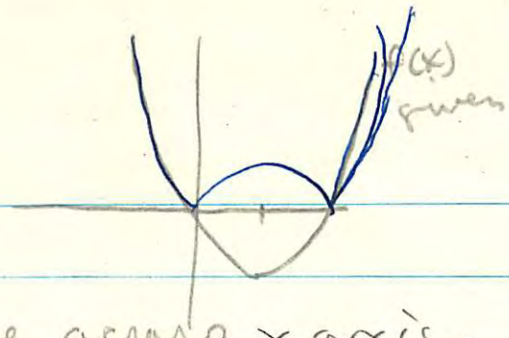
#15
(6)

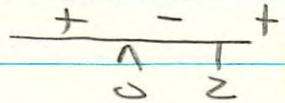


$f(x) = x^2 - 2x$

a) Just flip the neg portion of curve across x axis.

b) Deriv does not exist at $x=0$ because There is a sharp corner there. (This does not prove the fact – it only is used to establish the reason testing to prove/non existence. We must show the deriv does not exist)

$f(x) = x(x-2)$



$0 < x < 2$

$|f(x)| = f(x) \quad \forall x < 0$

$|f(x)| = -f(x) = 2x - x^2$

$f(x) = x^2 - 2x$

$f'(x) = 2x - 2$

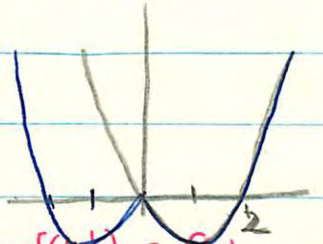
$f'(x) = 2 - 2x$

$\lim_{x \to 0^-} f'(x) = -2$

$\lim_{x \to 0^+} 2 - 2x = +2$

Since the left & Rt limit $\neq$, Deriv does not exist.

c)



ie $f(-x) = f(x)$ so reflects across y axis.

$f|x| = x^2 - 2|x|$

if $x < 0$

$f(x) = x^2 + 2x$

$f'(x) = 2x + 2$

$\lim_{x \to 0^-} f'(x) = 2$

if $x > 0$

$f(x) = x^2 - 2x$

$f'(x) = 2x - 2$

$\lim_{x \to 0^+} f'(x) = -2$

To be continuous $\lim_{x \to 0} f(|x|) = f(0)$

$\lim_{x \to 0^+} f(|x|) = \lim_{x \to 0^+} f(x) = 0$

$\lim_{x \to 0^-} f(|x|) = \lim_{x \to 0^-} f(-x) = 0$

Since (L) & (R) limits exist and are =, $f(|x|)$ is continuous.

Again, derivative does not exist since The left hand & Rt hand limits are not equal,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 - 2|x|}{x} \begin{cases} \lim_{x \to 0^+} x - 2 = -2 \\ \lim_{x \to 0^-} x + 2 = 2 \end{cases}$$

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AB
(7)

$f(x) = x^3 + ax^2 + bx + c$

i) f has PI at $(0, -2)$

ii) average of f is -3 for $x \in [0, 2]$

$f(0) = -2 \Rightarrow c = -2$

$f''(0) = 0 \qquad f' = 3x^2 + 2ax + b$

$f'' = 6x + 2a \qquad f''(0) = 0 \therefore a = 0$

So for $f(x) = x^3 + bx - 2$

$$-3 = \frac{1}{2}\int_0^2 x^3 + bx - 2 \, dx$$

$$-3 = \frac{1}{2}\left[\frac{x^4}{4} + \frac{bx^2}{2} - 2x\right]\Big|_0^2$$

a) $-6 = 4 + 2b - 4 \qquad b = -3 \qquad a = 0, b = -3, c = -2$

$f(x) = x^3 - 3x - 2.$

b) $f(0) = -2 \qquad f(3) = 27 - 9 - 2 = 16.$

$$\frac{16 + 2}{3 - 0} = \frac{18}{3} \overset{\text{set}}{=} f'(x) = 3x^2 - 3 = 6$$

$x^2 = 3$

$x = \pm\sqrt{3}$

reject $-\sqrt{3}$

$x = \sqrt{3}$

at $x = \sqrt{3}$ the tangent line to f has the same slope as the line passing through $(0, f(0))$ and $(3, f(3))$.

Mean value Thm: $\exists\ \underline{x} \ni$ m of tangent at $(\underline{x}, f(\underline{x}))$ which is // to the chord joining $(a, f(a))$ and $(b, f(b))$.

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Advanced Placement Examination

CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Given the function f defined by $f(x) = x^3 - x^2 - 4x + 4$.

   (a) Find the zeros of f.

   (b) Write an equation of the line tangent to the graph of f at $x = -1$.

   (c) The point (a, b) is on the graph of f and the line tangent to the graph at (a, b) passes through the point (0, −8) which is not on the graph of f. Find the values of a and b.

---

2. Let $f(x) = (1 - x)^2$ for all real numbers x, and let $g(x) = \ln x$ for all $x > 0$. Let $h(x) = (1 - \ln x)^2$.

   (a) Determine whether h(x) is the composition f(g(x)) or the composition g(f(x)).

   (b) Find $h'(x)$.

   (c) Find $h''(x)$.

   (d) On the axes provided, sketch the graph of h.

---

3. Given the function f defined by $f(x) = \dfrac{2x - 2}{x^2 + x - 2}$.

   (a) For what values of x is f(x) discontinuous?

   (b) At each point of discontinuity found in part (a), determine whether f(x) has a limit and, if so, give the value of the limit.

   (c) Write an equation for each vertical and horizontal asymptote to the graph of f. Justify your answer.

   (d) A rational function $g(x) = \dfrac{a}{b + x}$ is such that $g(x) = f(x)$ wherever f is defined. Find the values of a and b.

---

4. A particle moves on the X-axis so that its velocity at any time t is given by $v(t) = \sin 2t$. At $t = 0$, the particle is at the origin.

   (a) For $0 \leq t \leq \pi$, find all values of t for which the particle is moving to the left.

   (b) Write an expression for the position of the particle at any time t.

   (c) For $0 \leq t \leq \dfrac{\pi}{2}$, find the average value of the position function determined in part (b).

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5. Given the curve $x^2 - xy + y^2 = 9$.

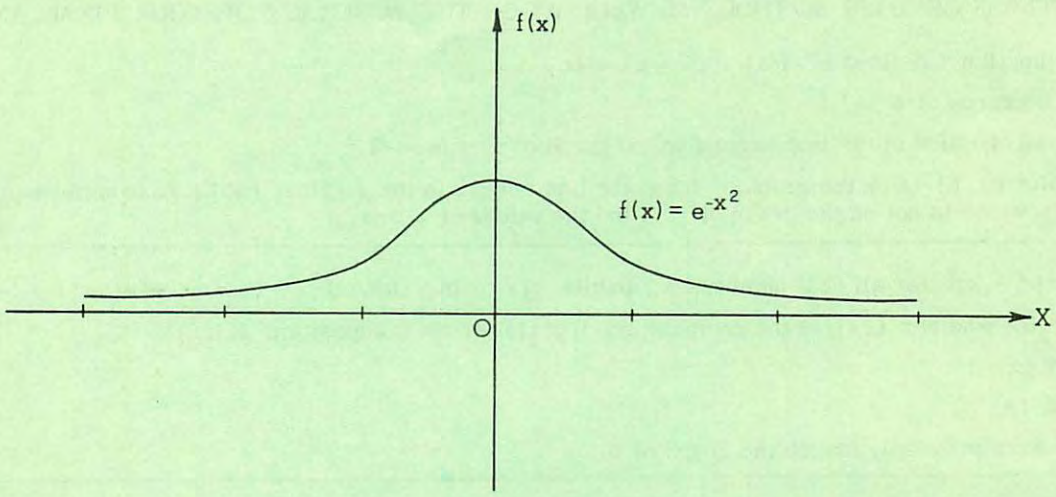
   (a) Write a general expression for the slope of the curve.

   (b) Find the coordinates of the points on the curve where the tangents are vertical.

   (c) At the point (0, 3) find the rate of change in the slope of the curve with respect to x.

---

6. Given the function f defined by $f(x) = e^{-x^2}$.



   (a) Find the maximum area of a rectangle that has two vertices on the X-axis and two on the graph of f. Justify your answer.

   (b) Let R be the region in the first quadrant bounded by the X- and Y-axes, the graph of f, and the line $x = k$. Find the volume of the solid generated by revolving R about the Y-axis.

   (c) Evaluate the limit of the volume determined in part (b) as k increases without bound.

---

7. Let g and h be any two twice-differentiable functions that are defined for all real numbers and that satisfy the following properties for all x:

   (i) $(g(x))^2 + (h(x))^2 = 1$

   (ii) $g'(x) = (h(x))^2$

   (iii) $h(x) > 0$

   (iv) $g(0) = 0$

   (a) Justify that $h'(x) = -g(x)h(x)$ for all x.

   (b) Justify that h has a relative maximum at $x = 0$.

   (c) Justify that the graph of g has a point of inflection at $x = 0$.

END OF EXAMINATION

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Advanced Placement Examination

CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Given the curve $x^2 - xy + y^2 = 9$.

   (a) Write a general expression for the slope of the curve.

   (b) Find the coordinates of the points on the curve where the tangents are vertical.

   (c) At the point (0, 3) find the rate of change in the slope of the curve with respect to x.

---

2. Given the function f defined by $f(x) = x|x + 2|$ for all x such that $-3 \leq x \leq 1$.

   (a) Find the values of x in the given interval for which f is increasing. Justify your answer.

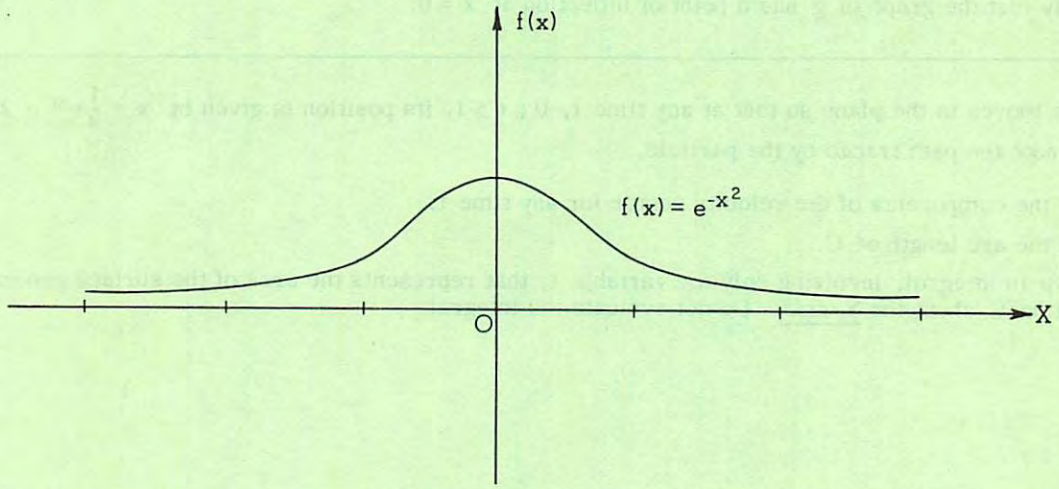
   (b) For what values of x is the graph of f concave downward?

   (c) On the axes provided, sketch the graph of f.

   (d) Is $f'(x)$ continuous for all x in the given interval? Justify your answer.

---

3. Given the function f defined by $f(x) = e^{-x^2}$



   (a) Find the maximum area of a rectangle that has two vertices on the X-axis and two on the graph of f. Justify your answer.

   (b) Let R be the region in the first quadrant bounded by the X- and Y-axes, the graph of f, and the line $x = k$. Find the volume of the solid generated by revolving R about the Y-axis.

   (c) Evaluate the limit of the volume determined in part (b) as k increases without bound.

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4. (a) In the interval $0 < x < \frac{\pi}{2}$ find the general solution of the differential equation $(\cot x)\frac{dy}{dx} + y = \csc x$.

   (b) Find the solution of the differential equation in part (a) that satisfies the condition that $y = 0$ when $x = \frac{\pi}{3}$.

---

5. The power series $\sum_{n=0}^{\infty} \frac{\ln(n+1)}{n+1} x^n$ has the interval of convergence $-1 \leq x < 1$. Let $f(x)$ be its sum.

   (a) Find $f(0)$ and $f'(0)$.

   (b) Justify that the interval of convergence is $-1 \leq x < 1$.

---

6. Let g and h be any two twice-differentiable functions that are defined for all real numbers and that satisfy the following properties for all x.

   (i) $(g(x))^2 + (h(x))^2 = 1$

   (ii) $g'(x) = (h(x))^2$

   (iii) $h(x) > 0$

   (iv) $g(0) = 0$

   (a) Justify that $h'(x) = -g(x)h(x)$ for all x.

   (b) Justify that h has a relative maximum at $x = 0$.

   (c) Justify that the graph of g has a point of inflection at $x = 0$.

---

7. A particle moves in the plane so that at any time t, $0 \leq t \leq 1$, its position is given by $x = \frac{1}{4}e^{8t} - 2t$ and $y = e^{4t}$. Let C denote the path traced by the particle.

   (a) Find the components of the velocity vector for any time t.

   (b) Find the arc length of C.

   (c) Set up an integral, involving only the variable t, that represents the area of the surface generated by rotating C about the Y-axis. Do not evaluate the integral.

END OF EXAMINATION

① $f(x) = x^3 - x^2 - 4x + 4$

$f(x) = (x-1)(x-2)(x+2)$

a) $f(x) = 0 \quad x \in \{1, \pm 2\}$

	1	−1	−4	4
1 ⌋	1	0	−4	0
−1 ⌋	1	−2	−2	6 ⇒ (−1, 6)
−1 ⌋	1	−3	1 = m	

b) $y - 6 = 1(x+1)$

$y = x + 7$

c) $f'(x) = 3x^2 - 2x - 4$

$f'(a) = 3a^2 - 2a - 4$

$m = \frac{b+6}{a}$ ⇐

$f(a) = b = a^3 - a^2 - 4a + 4$

$\frac{a^3 - a^2 - 4a + 12}{a} = 3a^2 - 2a - 4$

$a^3 - a^2 - 4a + 12 = 3a^3 - 2a^2 - 4a$

$2a^3 - a^2 - 12 = 0$

	2	−1	0	−12
2 ⌋	2	3	6	0

$a = 2$ only solution

$2x^2 + 3x + 6 = 0$

$x = \frac{-3 \pm \sqrt{9 - 48}}{}$

$b = 2^3 - 2^2 - 4(2) + 4$

$8 - 4 - 8 + 4 \Rightarrow b = 0$

$x \in \mathbb{C}$ ∴ no real Roots,

pt on curve (a, b) is $(2, 0)$

1978
(2)
AB

(2) $f(x) = (1-x)^2 \quad \forall x \in \mathbb{R}$

$g(x) = \ln x \quad \forall x > 0$

$h(x) = (1-\ln x)^2$

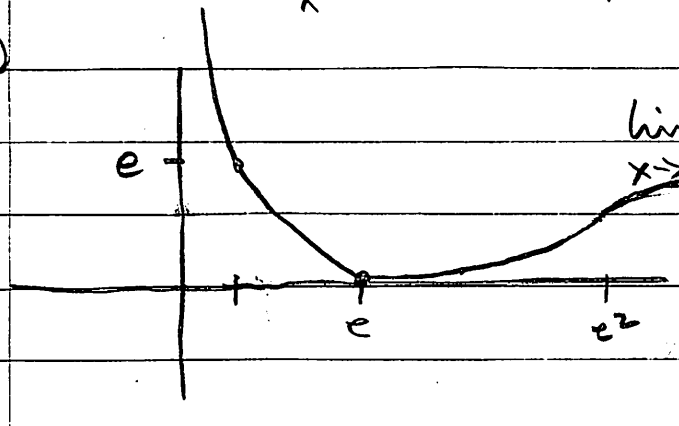
a) $f(g(x)) = f(\ln x) = (1-\ln x)^2$ so h is $f(g(x))$

b) $h'(x) = 2(1-\ln x)\left(-\frac{1}{x}\right) = -\frac{2}{x}(1-\ln x) = -2x^{-1}(1-\ln x)$

c) $h''(x) = (1-\ln x)(2x^{-2}) + \left(\frac{-2}{x}\left(\frac{-1}{x}\right)\right)$

$= \frac{2}{x^2}(1-\ln x) + \frac{2}{x^2} = \frac{2}{x^2}(2-\ln x)$

d)



$\lim_{x \to 0^+} (1-\ln x)^2 \Rightarrow +\infty$

$h'(1) = 0, \; h''(1) > 0 \therefore$ min at $(1, 0)$

$h''(x) = \frac{2}{x^2}(2-\ln x)$ $\quad 2 - \ln x > 0$

$\Downarrow$ $\quad \ln x < 2$

$+$ $\quad h''(x) > 0 \; \forall x < e^2$

$h''(x) < 0 \; \forall x > e^2$

$h(e^2) = (1-2)^2 = 1$ $\quad \lim_{x \to \infty} h(x) = +\infty$ (slowly)

1978
AB
(3)

$$f(x) = \frac{2x-2}{x^2+x-2} = \frac{2(x-1)}{(x+2)(x-1)} = \frac{2}{x+2}$$

a) discontinuous at $x \in \{-2, 1\}$.

b) $\lim_{x \to -2} \frac{2(x-1)}{(x+2)(x-1)} = \lim_{x \to -2} \frac{2}{(x+2)}$ - no limit.

$\lim_{x \to 1} \frac{2}{x+1} = \frac{2}{3}$

c) vertical asymptote at $x = -2$ restricts on Domain
(none at $x=1$ just open pt)
horizontal asymptote at $y=0$ since this
is the only Range restriction on $f(x)$.

d) $g(x) = \frac{a}{b+x}$ this is the fctn used in b above.

$a = b = 2$,

notice: $f(x) = \frac{2}{2+x}$ $\forall x$ where x is defined
so $g(x) = \frac{2}{2+x}$.

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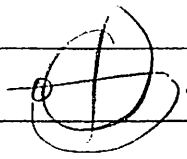
A15
4

(4) $v(\tau) = \sin 2\tau \qquad s(0) = 0$

(a) $0 \le \tau \le \pi$ find τ: $v(t) < 0$

$\sin 2\tau < 0$



$\pi < 2\tau < 2\pi$

$\frac{\pi}{2} < \tau < \pi$

b) $s(\tau) = \int v(\tau)\, d\tau = \int \sin 2\tau \, d\tau = -\frac{1}{2}\cos 2\tau + C$

$s(0) = 0 = -\frac{1}{2} + C \quad \therefore \quad C = \frac{1}{2}$

$s(\tau) = -\frac{1}{2}\cos 2\tau + \frac{1}{2}$

c) $\frac{1}{\frac{\pi}{2} - 0} \int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos 2\tau + \frac{1}{2}\, d\tau = \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} -\cos 2\tau + 1 \, d\tau =$

$\frac{1}{\pi}\left[-\frac{1}{2}\sin 2\tau + \tau\right]\Big|_0^{\frac{\pi}{2}} = \frac{1}{\pi}\left[0 + \frac{\pi}{2} - 0\right] = \frac{1}{2}$

1978
AB
(5)

$x^2 - xy + y^2 = 9$

a) find $\frac{dy}{dx}$ $\Rightarrow$ $2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$ $\quad u$ $\quad v$

b) Vertical tangents $\Rightarrow$ $\frac{dy}{dx} = \infty$ ie $\frac{dy}{dx}$ undefined.

when $2y - x = 0$ $\quad x = 2y$

$(2y)^2 - y(2y) + y^2 = 9$

$4y^2 - 2y^2 + y^2 = 9$

$3y^2 = 9$

$y = \pm\sqrt{3}$

$x^2 - x\sqrt{3} + 3 = 9$

$x^2 - x\sqrt{3} - 6 = 0$

$x = \frac{\sqrt{3} \pm \sqrt{3 + 24}}{2} = \frac{\sqrt{3} \pm 3\sqrt{3}}{2}$

$x^2 + x\sqrt{3} - 6 = 0$

$x = \frac{-\sqrt{3} \pm \sqrt{3 + 24}}{2}$

$x = \frac{-\sqrt{3} \pm 3\sqrt{3}}{2}$

$(2\sqrt{3}, \sqrt{3})$ $(-\sqrt{3}, \sqrt{3})$

$(-\sqrt{3}, -\sqrt{3})$ $(-2\sqrt{3}, -\sqrt{3})$

c) $\left.\frac{d^2y}{dx^2}\right|_{\substack{x=0 \\ y=3}}$ $\qquad \frac{d^2y}{dx^2} = \frac{(2y - x)\left(\frac{dy}{dx} - 2\right) - (y - 2x)\left(2\frac{dy}{dx} - 1\right)}{(2y - x)^2}$

$\left.\frac{dy}{dx}\right|_{\substack{x=0 \\ y=3}} = \frac{3 - 0}{6 - 0} = \frac{1}{2}$

$\frac{d^2y}{dx^2} = \frac{(6 - 0)\left(\frac{1}{2} - 2\right) - (3 - 0)\left(2 \cdot \frac{1}{2} - 1\right)}{(6 - 0)^2}$

$= \frac{6\left(-\frac{3}{2}\right) - 3(0)}{6^2} = -\frac{9}{36} = -\frac{1}{4}$

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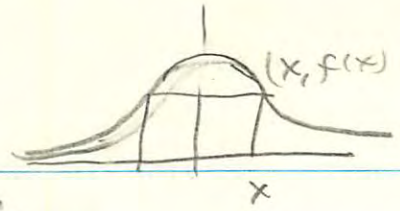
AB
6

$f(x) = e^{-x^2}$



a) f is symmetric to y axis

$A = 2x e^{-x^2}$

$A' = 2e^{-x^2} - 2x(2x)e^{-x^2}$

$2e^{-x^2} - 4x^2 e^{-x^2}$

$2e^{-x^2}(1 - 2x^2) \overset{\text{set}}{=} 0$

$2x^2 = 1$

$x = \pm \frac{\sqrt{2}}{2}$

$A'' = -4x e^{-x^2}(1-2x^2) - 2e^{-x^2}(-4x)$

$A'|_{x = \frac{\sqrt{2}}{2} - \varepsilon}$ +

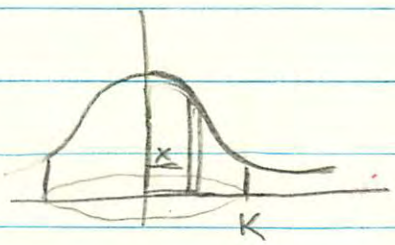
$A'|_{x = \frac{\sqrt{2}}{2} - \varepsilon}$ −

∴ max by f' test

$A|_{x = \frac{\sqrt{2}}{2}} = 2\frac{\sqrt{2}}{2} e^{-\frac{1}{2}}$

$\frac{2\sqrt{2}}{2e^{\frac{1}{2}}} = \frac{\sqrt{2}}{e^{\frac{1}{2}}}$

b)



$2\pi \int_0^K$ rad · height dx

$2\pi \int_0^K x e^{-x^2} dx$ let $u = -x^2$ $x=0\ u=0$

$du = -2x\,dx$ $x=K\ u=-K^2$

$\frac{1}{2} 2\pi \int_{-K^2}^{0} e^u du = \pi e^u \Big|_{-K^2}^{0} = \pi - \pi e^{-K^2}$

c) $\lim_{K \to \infty} \pi\left(1 - \frac{1}{e^{K^2}}\right) = \pi(1-0) = \pi$

1978
AB
7

Given i) $g^2 + h^2 = 1$

ii) $g' = h^2$

iii) $h > 0$

iv) $g(0) = 0$

a) $h'(x) = -g(x)\,h(x) \quad \forall x$

i $g^2 + h^2 = 1$

$2g g' + 2h h' = 0$

ii $2g h^2 + 2h h' = 0 \quad h > 0$ ie $h \neq 0$

$gh + h' = 0$

$h'(x) = -g(x)\,h(x)$

b) h has rel max at $x=0$ this requires $h'(0) = 0$, $h''(0) < 0$

by "a" $h'(0) = -g(0)\,h(0)$ by (iv) $g(0) = 0$

$h''(x) = -h(x)\,g'(x) - g(x)\,h'(x)$

$h''(0) = -h(0)\,g'(0) - g(0)\,h'(0)$ ($g(0)\,h'(0)$: 0 by iv)

$-h(0)\,(h^2(0))$

$-h^3(0)$

but $h(x) > 0 \; \forall x$ ∴

$h''(0) < 0$, ∴ h has rel max at $x=0$

c) g has PI at $x=0 \Rightarrow g''(0) = 0$

ii $g' = h^2 \qquad g'' = 2h h'$

$= 2h(-g(x)\,h(x))$

$g''(0) = 2(+)(-(0)+) = 0$

1977

College Entrance Examination Board

Advanced Placement Examination

MATHEMATICS: CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Let $f(x) = \cos x$ for $0 \leq x \leq 2\pi$, and let $g(x) = \ln x$ for all $x > 0$. Let S be the composition of g with f, that is, $S(x) = g(f(x))$.

   (a) Find the domain of S.

   (b) Find the range of S.

   (c) Find the zeros of S.

   (d) Find the slope of the line tangent to the graph of S at $x = \frac{\pi}{3}$.

---

2. Consider the function f defined by $f(x) = (x^2 - 1)^3$ for all real numbers x.

   (a) For what values of x is the function increasing?

   (b) Find the x- and y-coordinates of the relative maximum and minimum points. Justify your answer.

   (c) For what values of x is the graph of f concave upward?

   (d) Using the information found in parts (a), (b), and (c), sketch the graph of f on the axes provided.

---

3. Given the function f defined for all real numbers x by $f(x) = e^{\frac{x}{2}}$.

   (a) Find the area of the region R bounded by the line $y = e$, the graph of f, and the Y-axis.

   (b) Find the volume of the solid generated by revolving R, the region in (a), about the X-axis.

---

4. Let f and g and their inverses f^{-1} and g^{-1} be differentiable functions and let the values of f, g, and the derivatives f′ and g′ at $x = 1$ and $x = 2$ be given by the table below:

   Determine the value of each of the following.

x	f(x)	g(x)	f′(x)	g′(x)
1	3	2	5	4
2	2	π	6	7

   (a) The derivative of $f + g$ at $x = 2$

   (b) The derivative of fg at $x = 2$

   (c) The derivative of $\frac{f}{g}$ at $x = 2$

   (d) $h'(1)$ where $h(x) = f(g(x))$

   (e) The derivative of g^{-1} at $x = 2$

5. A particle moves along the X-axis with acceleration given by $a(t) = 2t - 10 + \frac{12}{t}$ for $t \geq 1$.

   (a) Write an expression for the velocity $v(t)$, given that $v(1) = 9$.

   (b) For what values of t, $1 \leq t \leq 3$, is the velocity a maximum? Justify your answer.

   (c) Write an expression for the position $x(t)$, given that $x(1) = -16$.

---

6. A rectangle has a constant area of 200 square meters and its length L is increasing at the rate of 4 meters per second.

   (a) Find the width W at the instant the width is decreasing at the rate of 0.5 meters per second.

   (b) At what rate is the diagonal D of the rectangle changing at the instant when the width W is 10 meters?

---

7. Let f be the real-valued function defined by $f(x) = \sin^3 x + \sin^3 |x|$.

   (a) Find $f'(x)$ for $x > 0$.

   (b) Find $f'(x)$ for $x < 0$.

   (c) Determine whether $f(x)$ is continuous at $x = 0$. Justify your answer.

   (d) Determine whether the derivative of $f(x)$ exists at $x = 0$. Justify your answer.

END OF EXAMINATION

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College Entrance Examination Board

Advanced Placement Examination

MATHEMATICS: CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

1. Let f be the function defined by $f(x) = \frac{x}{1+x^2}$ for all real numbers x.

   (a) (i) Find the zeros of f.

   (ii) Investigate the symmetry of the graph of f with respect to the X-axis, the Y-axis, and the origin.

   (iii) Find the x- and y-coordinates of the relative maximum and minimum points of f. Justify your answer.

   (iv) Describe the behavior of the graph of f for large $|x|$.

   (v) Using the information found above, sketch the graph of f on the axes provided.

   (b) Find the area of the region bounded by the graph of f, the X-axis, and the lines $x = \frac{1}{k}$ and $x = k$ for $k > 1$.

---

2. Let f and g and their inverses f^{-1} and g^{-1} be differentiable functions and let the values of f, g, and the derivatives f′ and g′ at $x = 1$ and $x = 2$ be given by the table below.

   Determine the value of each of the following.

   (a) The derivative of $f + g$ at $x = 2$

   (b) The derivative of fg at $x = 2$

   (c) The derivative of $\frac{f}{g}$ at $x = 2$

   (d) $h'(1)$ where $h(x) = f(g(x))$

   (e) The derivative of g^{-1} at $x = 2$

x	f(x)	g(x)	f′(x)	g′(x)
1	3	2	5	4
2	2	π	6	7

---

3. A particle moves along the X-axis with acceleration given by $a(t) = 2t - 10 + \frac{12}{t}$ for $t \geq 1$.

   (a) Write an expression for the velocity v(t), given that $v(1) = 9$.

   (b) For what values of t, $1 \leq t \leq 3$, is the velocity a maximum? Justify your answer.

   (c) Write an expression for the position x(t), given that $x(1) = -16$.

4. (a) If $\sqrt{2}e^{2x}$ is a particular solution of the differential equation $y'' - 4y' + py = 0$, find the value of p and the general solution of the equation.

   (b) Find the particular solution of the differential equation $y'' - 4y' + 5y = 0$ that satisfies the condition that $y = 0$ and $y' = 1$ when $x = \frac{\pi}{2}$.

---

5. (a) Does the series $\sum_{j=1}^{\infty} \frac{2}{j^2} \sin\left(\frac{\pi}{j}\right)$ converge? Justify your answer.

   (b) Express $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{n} \sin\left(\frac{k\pi}{n}\right)$ as a definite integral and evaluate the integral.

---

6. Let f be the real-valued function defined by $f(x) = \sin^3 x + \sin^3 |x|$.

   (a) Find $f'(x)$ for $x > 0$.

   (b) Find $f'(x)$ for $x < 0$.

   (c) Determine whether $f(x)$ is continuous at $x = 0$. Justify your answer.

   (d) Determine whether the derivative of $f(x)$ exists at $x = 0$. Justify your answer.

---

7. Let $F(x) = \int_0^x \frac{1}{1+t^4}\,dt$ for all real numbers x.

   (a) Find F(0).

   (b) Find F′(1).

   (c) Justify that $F(3) - F(1) < 1$.

   (d) Justify that $F(x) + F(-x) = 0$ for all real numbers x.

END OF EXAMINATION

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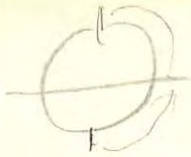
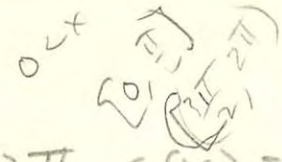




1977
AB
①

① $f(x) = \cos x \quad 0 \le x \le 2\pi \qquad g(x) = \ln x \quad x > 0$

$S = g(f(x))$

$D_f \quad x \in \mathbb{R} \; 0 \le x \le 2\pi \qquad D_g \quad x > 0$

$R_f \quad |f(x)| \le 1 \qquad R_g \quad g(x) \in \mathbb{R}.$

$D_S \quad x \in R \quad 0 < x \le \frac{\pi}{2}, \; \frac{3\pi}{4} < x \le 2\pi$

$R_S \quad \text{since } 0 < f(x) \le 1 \quad g(f(x)) \le 0 \text{ as } x \to 0 \text{ or}$
$x \to \frac{3\pi}{4} \quad f(x) \to 0 \quad g(f(x)) \to -\infty$

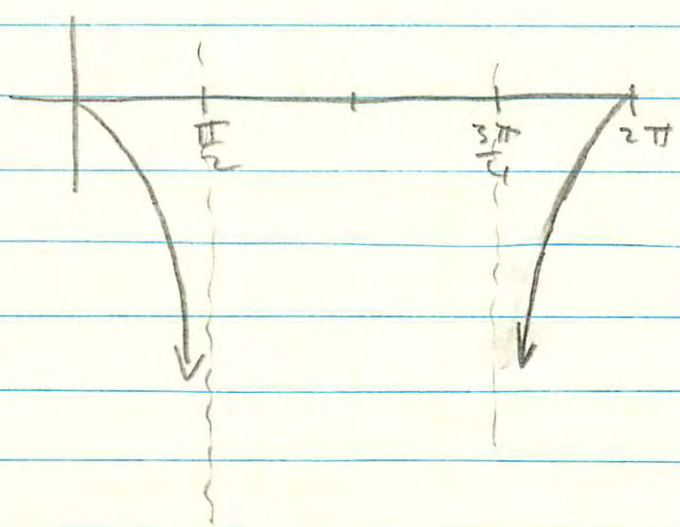
$S = 0$ when $f = 1$ $\quad f = 1$ when $\cos x = 1$ i.e.

$S = 0$ when $x \in \{0, 2\pi\}$.

$S = \ln(\cos x)$

$S' = \frac{1}{\cos x}(-\sin x) = -\tan x \Big|_{\frac{\pi}{3}} = -\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}}{3}.$

$S'' < 0 \; \forall x \; \therefore$ Concave down



1977
AB
2

2 $f(x) = (x^2-1)^3 \quad \forall x \in \mathbb{R}$,

$f'(x) = 3(x^2-1)^2(2x)$

$f'(x) > \quad \forall x > 0$

a) $f''(x) < \quad \forall x < 0 \quad \therefore$ f is increasing when $x > 0$

b) $f'(x) = 6x(x^2-1)^2 \overset{\text{Set}}{=} 0$

$x \in \{\pm 1, 0\}$

$f(\pm 1) = 0 \quad f(0) = -1$

$x > 1$	+	+	+
$0 < x < 1$	+	+	+
$-1 < x < 0$	−	+	−
	−	+	−

No max only min $(0, -1)$

c)

$f'' = 6(x^2-1)^2 + (6x)(2x)\,2(x^2-1)$

$= 6(x^2-1)(x^2-1+4x^2)$

$= 6(x^2-1)(5x^2-1) \overset{\text{Set}}{>} 0$

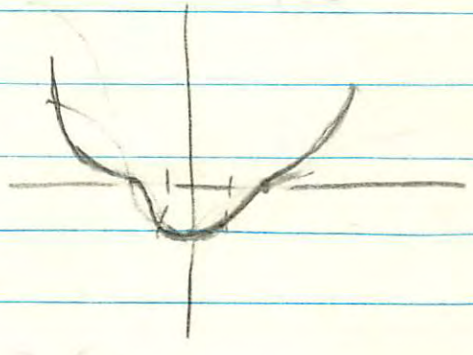
$|x| < 1$

Concave up

$|x| < \frac{1}{5}$

$|x| > 1$

			−
f'	−	0	+
		0	1



$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$$(g)'(g^{-1}(x)) = \frac{1}{(g^{-1})'(x)} \leftarrow \text{what we want.}$$

$$(g)'(g^{-1}(2)) = \frac{1}{(g^{-1})'(2)}$$

But

~~$g'(x) = 1$~~ ∴

~~g^{-1}~~

Since $g(1) = 2$

then $g^{-1}(2) = 1$

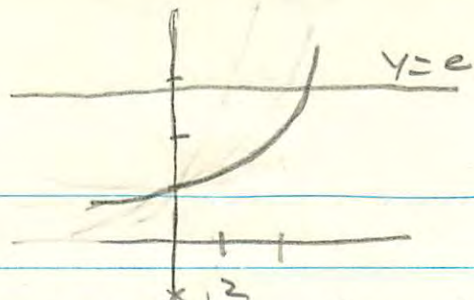
$$g'(1) = \frac{1}{(g^{-1})'(2)}$$

$$4 = \frac{1}{(g^{-1})'(2)}$$

$$(g^{-1})'(2) = \frac{1}{4}$$

∴

1977
AB
③ ④



③ $f(x) = e^{\frac{x}{2}}$

a) $\int_0^2 e - e^{\frac{x}{2}}\,dx = ex - 2e^{\frac{x}{2}}\Big|_0^2 = 2e - 2e - 0 + 2 = 2$

b) $\pi\int_0^2 e^2 - e^x\,dx = \pi\left(xe^2 - e^x\right)\Big|_0^2 =$ (washers)

$\pi(2e^2 - e^2 + 1) = \pi(e^2+1)$

④ a) $(f+g)' = f' + g'\Big|_{x=2} = 6 + 7 = 13$

b) $(fg)' = fg' + gf'\Big|_{x=2} = 2\cdot 7 + 6\pi = 14 + 6\pi$

c) $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}\Big|_{x=2} = \frac{6\pi - 2\cdot 7}{\pi^2} = \frac{6\pi - 14}{\pi^2}$

Pg 137 d) $h'(1) = ?$ $f(g(x)) = h(x)$

$h'(x) = f'(g(x))\cdot g'(x)$ Chain rule

$f'(2) \cdot 4$

$6 \cdot 4 = 24$

Pg 147 e) $\frac{D(g^{-1})}{dx}\Big|_{x=2}$ $f(x) = \frac{1}{g'(x)(g^{-1}(x))'}$ So

$\left(g^{-1}(x)\right)'\Big|_{x=2} = \frac{1}{g(x)g'(x)}$

$= \frac{1}{\pi\cdot 7} = \frac{1}{7\pi}$

$\frac{1}{4}$

Pg 220 of ANALysis we want $(g^{-1})'(2) = A'(2)$ So $B = g$

$A'(x) = \frac{1}{B'(A'x)}$ where A & B and inverse fcns

$(g^{-1})'(x) = \frac{1}{g'(g^{-1}(2))} = \frac{1}{g'(7)}$

1977
(5)
AB

(5) $a(t) = 2t - 10 + \frac{12}{t} \quad t \geq 1$

a) $v(t) = t^2 - 10t + 12\ln t + C$

$v(1) = 9 = t^2 - 10t + 12\ln t + C\Big|_{t=1} \Rightarrow$

$9 = 1 - 10 + 0 + C \Rightarrow C = 18$

(a) $\longrightarrow v(t) = t^2 - 10t + 12\ln t + 18$

1st der of veloc is $a(t) = 2t - 10 + \frac{12}{t} \overset{\text{set}}{=} 0$

$v''(t) = 2 - \frac{12}{t^2}$

$v''(2) \leq 0$ ∴ max. at $t = 2$

$v''(3) \geq 0$ ∴ min.

(b) $\longrightarrow 2t^2 - 10t + 12 = 0$

$t^2 - 5t + 6 = 0$

$(t-3)(t-2) = 0$

$t = 3 \quad t = 2$

max at $t = 2$

$x(t) = \int v(t)\,dt = \int t^2 - 10t + 12\ln t + 18\,dt =$

let $u = \ln t \quad v = t$

$du = \frac{1}{t}\,dt \quad dv = dt$

$\frac{1}{3}t^3 - 5t^2 + 18t + 12\int \ln t\,dt$

$x(t) = \frac{1}{3}t^3 - 5t^2 + 18t + 12t\ln t - 12t + C$

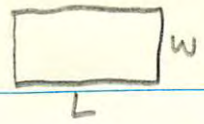
$x(1) = -16 = \frac{1}{3} - 5 + 18 + 0 - 12 + C$

$-16 = \frac{4}{3} + C$

$-\frac{52}{3} = C$

$x(t) = \frac{1}{3}t^3 - 5t^2 + 18t + 12t\ln t - 12t - \frac{52}{3}$

1977
AB
6

6 

$\frac{dL}{dt} = 4/\text{sec}$

$wL = 200$

a) $w\frac{dL}{dt} + L\frac{dw}{dt} = 0$

when width is decreasing at $\frac{1}{2}$ meter/sec the width is 5 meter

$4w + \frac{1}{2}L = 0$

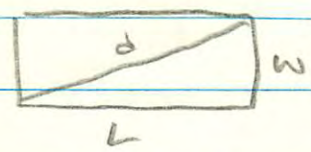
$8w - L = 0$

$8w - \frac{200}{w} = 0$

$8w^2 - 200 = 0$

$8(w^2 - 25) = 0$

$w = 5$

try of b) 

$100 + 400 \quad \therefore d = 10\sqrt{5}$

$\frac{dd}{dt} = ?$ $\quad d^2 = w^2 + L^2$

$d^2 = w^2 + \left(\frac{200}{w}\right)^2$

$\frac{dw}{dt}$? $\quad 2d\frac{dd}{dt} = 2w - 200(2w)^{-3}$

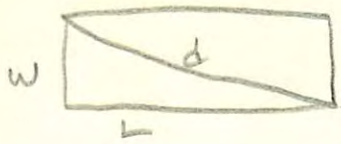
$\frac{dd}{dt} = \left.\frac{w - 200w^{-3}}{d}\right|_{\substack{w=10 \\ d=10\sqrt{5}}}$

$\frac{10 - \frac{200}{1000}}{10\sqrt{5}} = \frac{100-2}{100\sqrt{5}} = \frac{98\sqrt{5}}{5\cdot 100} = \frac{49\sqrt{5}}{250}$

$\cancel{2}d\frac{dd}{dt} = \cancel{2}w\frac{dw}{dt} + \cancel{2}L\frac{dL}{dt}$

$10\sqrt{5}\frac{dd}{dt} = 10$

6 B



$100 + 400 = 500$

$w = 10$ $\quad \frac{dL}{dt} = +4$

$L = 20$

1977

A B

6b

$wL = 200 \qquad d^2 = w^2 + L^2$

$$d^2 = \left(\frac{200}{L}\right)^2 + L^2 = (200)^2 L^{-2} + L^2$$

$$\cancel{2}\, d \frac{dd}{dt} = -\cancel{2}(200)^2 L^{-3} \frac{dL}{dt} + \cancel{2} L \frac{dL}{dt}$$

$$10\sqrt{5}\, \frac{dd}{dt} = -\frac{\cancel{200} \cdot \cancel{200}}{\cancel{8000}_{10}}(\cancel{4}) + 20(4)$$

$$10\sqrt{5}\, \frac{dd}{dt} = -20 + 80$$

$$\frac{dd}{dt} = \frac{6\cancel{0}}{1\cancel{0}\sqrt{5}} = \frac{6\sqrt{5}}{5}$$

1977
AB
(7)

7) $f(x) = \sin^3 x + \sin^3 |x|$

if $x > 0$ $\quad f(x) = 2\sin^3 x$

a) $\quad f'(x) = 6\sin^2 x \cos x$

b) if $x < 0$ $\quad f(x) = 0$

$f'(x) = 0$

c) if f is continuous at $x=0$ Then $f(0)$ exists.

and $f'(0) = \lim_{x\to 0} f(x)$

$f(0) = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 2\sin^3 x = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} 0 = 0$ $\quad \therefore$ f is continuous at $x=0$

d) $f'(0)$ Exist?

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ so $f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$

$f(0) = 0$ $\quad f(h) = \sin^3 h + \sin^3 |h|$

$$\lim_{h\to 0} \frac{\sin^3 h + \sin^3 |h|}{h} = \lim_{h\to 0} \left(\sin^2 h \left(\frac{\sin h}{h} \right) + \sin^2 |h| \frac{\sin |h|}{h} \right)$$

$\sin^2 h \to 0$, $\quad \sin^2 |h| \to 0$

$\lim_{\theta\to 0} \sin\theta = 0$ $\quad \therefore$ $\lim_{\theta\to 0^+} \sin\theta = \lim_{\theta\to 0^-} \sin\theta$ so $\lim_{|h|\to 0} \sin^2 |h| = 0$

College Entrance Examination Board

Advanced Placement Examination

MATHEMATICS: CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

1. Let f be the real-valued function defined by $f(x) = \sqrt{1 + 6x}$.

   (a) Give the domain and range of f.

   (b) Determine the slope of the line tangent to the graph of f at $x = 4$.

   (c) Determine the y-intercept of the line tangent to the graph of f at $x = 4$.

   (d) Give the coordinates of the point on the graph of f where the tangent line is parallel to $y = x + 12$.

---

2. Given the two functions f and h such that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \frac{f(x)}{x - 3}, & \text{for } x \neq 3 \\ p, & \text{for } x = 3. \end{cases}$

   (a) Find all zeros of the function f.

   (b) Find the value of p so that the function h is continuous at $x = 3$. Justify your answer.

   (c) Using the value of p found in (b), determine whether h is an even function. Justify your answer.

---

3. Let R be the region bounded by the curves $f(x) = \frac{4}{x}$ and $g(x) = (x - 3)^2$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated by revolving R about the X-axis.

---

4. (a) A point moves on the hyperbola $3x^2 - y^2 = 23$ so that its y-coordinate is increasing at a constant rate of 4 units per second. How fast is the x-coordinate changing when $x = 4$?

   (b) For what values of k will the line $2x + 9y + k = 0$ be normal to the hyperbola $3x^2 - y^2 = 23$?

---

5. Given the function defined by $y = e^{\sin x}$ for all x such that $-\pi \leq x \leq 2\pi$.

   (a) Find the x- and y-coordinates of <u>all</u> maximum and minimum points on the given interval. Justify your answers.

   (b) On the axes provided, sketch the graph of the function.

   (c) Write an equation for the axis of symmetry of the graph.

---

6. (a) Given $5x^3 + 40 = \int_c^x f(t)\,dt$.

   (i) Find $f(x)$.

   (ii) Find the value of c.

   (b) If $F(x) = \int_x^3 \sqrt{1 + t^{16}}\,dt$, find $F'(x)$.

---

7. For a differentiable function f, let f* be the function defined by $f^*(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{h}$.

   (a) Determine $f^*(x)$ for $f(x) = x^2 + x$.

   (b) Determine $f^*(x)$ for $f(x) = \cos x$.

   (c) Write an equation that expresses the relationship between the functions f* and f′, where f′ denotes the usual derivative of f.

END OF EXAMINATION

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College Entrance Examination Board

Advanced Placement Examination

MATHEMATICS: CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

1. A particle moves on the X-axis in such a way that its position at time t is given by $x = (2t - 1)(t - 1)^2$.

   (a) At what times t is the particle at rest? Justify your answer.

   (b) During what interval of time is the particle moving to the left? Justify your answer.

   (c) At what time during the interval found in (b) is the particle moving most rapidly (that is, the speed is a maximum)? Justify your answer.

---

2. Let R be the region bounded by the curves $f(x) = \frac{4}{x}$ and $g(x) = (x - 3)^2$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated by revolving R about the X-axis.

---

3. Given the function defined by $y = e^{\sin x}$ for all x such that $-\pi \leq x \leq 2\pi$.

   (a) Find the x- and y-coordinates of all maximum and minimum points on the given interval. Justify your answers.

   (b) On the axes provided, sketch the graph of the function.

   (c) Write an equation for the axis of symmetry of the graph.

---

4. (a) Determine $\int x^2 e^{5x}\, dx$.

   (b) Using integration by parts, derive a general formula for $\int x^n e^{kx}\, dx$, $k \neq 0$, in which the resulting integrand involves x^{n-1}.

---

5. (a) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 3y = 0$.

   (b) Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 3y = 1 - e^{-x}$.

   (c) Find the solution of the differential equation in (b) that satisfies the conditions that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$.

---

6. For a differentiable function f, let f* be the function defined by $f^*(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{h}$.

   (a) Determine $f^*(x)$ for $f(x) = x^2 + x$.

   (b) Determine $f^*(x)$ for $f(x) = \cos x$.

   (c) Write an equation that expresses the relationship between the functions f* and f′, where f′ denotes the usual derivative of f. Justify your answer.

---

7. (a) Write the first three nonzero terms and the general term of the Taylor series expansion about $x = 0$ of $f(x) = 5 \sin \frac{x}{2}$.

   (b) What is the interval of convergence for the series found in (a)? Show your method.

   (c) What is the minimum number of terms of the series in (a) that are necessary to approximate f(x) on the interval $(-2, 2)$ with an error not exceeding 0.1? Show your method.

END OF EXAMINATION

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College Entrance Examination Board

Advanced Placement Examination

MATHEMATICS: CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Given the function defined by $y = x + \sin x$ for all x such that $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$.

   (a) Find the coordinates of all maximum and minimum points on the given interval. Justify your answers.
   (b) Find the coordinates of all points of inflection on the given interval. Justify your answers.
   (c) On the axes provided, sketch the graph of the function.

---

2. Let R be the region in the first quadrant bounded by the graphs of $\frac{x^2}{9} + \frac{y^2}{81} = 1$ and $3x + y = 9$.

   (a) Set up but do not evaluate an integral representing the area of R. Express the integrand as a function of a single variable.
   (b) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the X-axis. Express the integrand as a function of a single variable.
   (c) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the Y-axis. Express the integrand as a function of a single variable.

---

3. A particle moves on the circle $x^2 + y^2 = 1$ so that at time $t \geq 0$ the position is given by the vector $\left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2}\right)$.

   (a) Find the velocity vector.
   (b) Is the particle ever at rest? Justify your answer.
   (c) Give the coordinates of the point that the particle approaches as t increases without bound.

---

4. (a) Determine whether the series

$$\frac{1}{3} - \frac{2^3}{3^2} + \frac{3^3}{3^3} - \frac{4^3}{3^4} + \ldots + \frac{(-1)^{n-1} n^3}{3^n} + \ldots$$

   is convergent. Justify your answer.

   (b) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$. Justify your answer.

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5. Given a function f with the following properties.

(i) $f(x+h) = e^h f(x) + e^x f(h)$ for all real numbers x and h.

(ii) $f(x)$ has a derivative for all real numbers x.

(iii) $f'(0) = 2$.

(a) Show that $f(0) = 0$.

(b) Using the definition of $f'(0)$, find $\lim_{x \to 0} \frac{f(x)}{x}$.

(c) Prove there exists a real number p such that $f'(x) = f(x) + pe^x$ for all real numbers x.

(d) What is the value of the number p that is described in (c) ?

---

6. Given the differential equation $y'' - (p+1)y' + py = 0$, where p is a real number <u>not</u> equal to 1.

(a) Find the general solution of the differential equation.
(b) Find the particular solution satisfying the conditions $y(0) = 0$ and $y'(0) = 1$.
(c) Find the limit of the solution in (b) as $p \to 1$. Justify your answer.

---

7. (a) For what value of m is the line $y = mx$ tangent to the graph of $y = \ln x$?
(b) Prove that the graph of $y = \ln x$ lies entirely below the graph of the line found in (a).
(c) Use the results of (b) to show that $e^x \geq x^e$ for $x > 0$.

END OF EXAMINATION

College Entrance Examination Board

Advanced Placement Examination

MATHEMATICS: CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Given the function f defined by $f(x) = \ln(x^2 - 9)$.
   (a) Describe the symmetry of the graph of f.
   (b) Find the domain of f.
   (c) Find all values of x such that $f(x) = 0$.
   (d) Write a formula for $f^{-1}(x)$, the inverse function of f, for $x > 3$.

---

2. A particle moves along the X-axis in such a way that its position at time t for $t \geq 0$ is given by $x = \frac{1}{3}t^3 - 3t^2 + 8t$.
   (a) Show that at time $t = 0$ the particle is moving to the right.
   (b) Find all values of t for which the particle is moving to the left.
   (c) What is the position of the particle at time $t = 3$?
   (d) When $t = 3$, what is the total distance the particle has traveled?

---

3. Given the function f defined for all real numbers x by $f(x) = 2|x - 1|x^2$.
   (a) What is the range of the function?
   (b) For what values of x is the function continuous?
   (c) For what values of x is the derivative of $f(x)$ continuous?
   (d) Determine the value of $I = \int_0^1 f(x)\,dx$.

---

4. Given the function defined by $y = x + \sin x$ for all x such that $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$.
   (a) Find the coordinates of <u>all</u> maximum and minimum points on the given interval. Justify your answers.
   (b) Find the coordinates of <u>all</u> points of inflection on the given interval. Justify your answers.
   (c) On the axes provided, sketch the graph of the function.

---

5. The line $x = c$ where $c > 0$ intersects the cubic $y = 2x^3 + 3x^2 - 9$ at point P and the parabola $y = 4x^2 + 4x + 5$ at point Q.
   (a) If a line tangent to the cubic at point P is parallel to the line tangent to the parabola at point Q, find the value of c where $c > 0$.
   (b) Write the equations of the two tangent lines described in (a).

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6. Let R be the region in the first quadrant bounded by the graphs of $\frac{x^2}{9} + \frac{y^2}{81} = 1$ and $3x + y = 9$.

(a) Set up but do not evaluate an integral representing the area of R. Express the integrand as a function of a single variable.

(b) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the X-axis. Express the integrand as a function of a single variable.

(c) Set up but do not evaluate an integral representing the volume of the solid generated when R is rotated about the Y-axis. Express the integrand as a function of a single variable.

---

7. Given a function f with the following properties.

(i) $f(x + h) = e^h f(x) + e^x f(h)$ for all real numbers x and h.

(ii) $f(x)$ has a derivative for all real numbers x.

(iii) $f'(0) = 2$.

(a) Show that $f(0) = 0$.

(b) Using the definition of $f'(0)$, find $\lim_{x \to 0} \frac{f(x)}{x}$.

(c) Prove there exists a real number p such that $f'(x) = f(x) + pe^x$ for all real numbers x.

(d) What is the value of the number p that is described in (c) ?

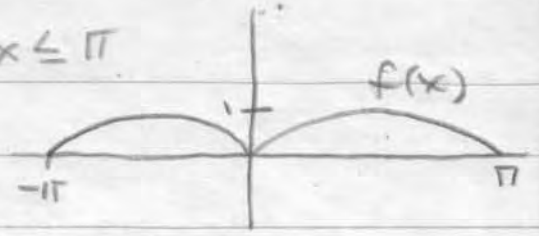
END OF EXAMINATION

AB 1974 AB

$g(x) = x^2$

(1) $f(x) = |\sin x| \quad -\pi \le x \le \pi$

graph is $\sin x$ then neg portions is rotated to positive



b) $H(x) = g(f(x)) = \left(|\sin x|\right)^2 = \sin^2 x$

c) $D_h \quad x \in \mathbb{R}$

$R_h \quad h(x) \in \mathbb{R} \quad 0 \le h(x) \le 1$

d) $H'(x) = 2\sin x \cos x = 2\sin^2 x \Big|_{x=\frac{\pi}{4}} = 2$

$h\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$y - \frac{1}{2} = 2\left(x - \frac{\pi}{4}\right)$

$y = 2x - \frac{\pi}{2} + \frac{1}{2}$

$y = 2x - \frac{1}{2}(\pi - 1)$

(2) $P(x) = x^4 + ax^3 + bx^2 + cx + d$ since this is symmetrical to y axis $P(x) = P(-x)$ so $a = c = 0$.

$P(x) = x^4 + bx^2 + d$. rel max at $(0,1) \Rightarrow$

$P'(0) = 0 \quad P(0) = 1 \quad P''(0) < 0$

$P'(0) = 1 \Rightarrow \quad d = 1$

$P'(x) = 4x^3 + 2bx \overset{\text{set}}{=} 0$

$2x(2x^2 + b) = 0 \quad x = 0$

$P''(\ ) = 12x^2 + 2b < 0$ at

$b < 0$

$P(q) = q^4 + bq^2 + 1 = -3$

$P'(q) = 4q^3 + 2bq \overset{\text{set}}{=} 0$

$2q(q^2 + b) = 0$

$P(q) = q^4 + bq^2 + 4 = 0$

$(q+2)^2 \quad \therefore b = -4.$

could be ± 2

but say $q = -2$

a) $P(x) = x^4 + 4x^2 + 1$

$P'(x) = 4x^3 + 8x \overset{\text{set}}{=} 0$

$4x(x^2 + 2) \quad 0$ $\quad x = 0$

$P(0) = 1$

$P''(x) = 12x^2 + 8 > 0 \quad x = 0$

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College Entrance Examination Board

Advanced Placement Examination

MATHEMATICS: CALCULUS AB

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Given $f(x) = |\sin x|$, $-\pi \leq x \leq \pi$, and $g(x) = x^2$ for all real x.

   (a) On the axes provided sketch the graph of f.

   (b) Let $H(x) = g(f(x))$. Write an expression for $H(x)$.

   (c) Find the domain and range of H.

   (d) Find an equation of the line tangent to the graph of H at the point where $x = \frac{\pi}{4}$.

---

2. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$. The graph of $y = P(x)$ is symmetric with respect to the Y-axis, has a relative maximum at (0, 1), and has an absolute minimum at (q, −3).

   (a) Determine the values of a, b, c, and d, and using these values write an expression for $P(x)$.

   (b) Find all possible values of q.

---

3. Let $f(x) = kx^2 + c$.

   (a) Find x_0 in terms of k such that the tangent lines to the graph of f at $(x_0, f(x_0))$ and $(-x_0, f(-x_0))$ are perpendicular.

   (b) Find the slopes of the tangent lines mentioned in (a).

   (c) Find the coordinates, in terms of k and c, of the point of intersection of the tangent lines mentioned in (a).

---

4. Let f be a function defined for all $x > -5$ and having the following properties.

   (i) $f''(x) = \frac{1}{3\sqrt{x+5}}$ for all x in the domain of f.

   (ii) The line tangent to the graph of f at (4, 2) has an angle of inclination of 45°.

   Find an expression for $f(x)$.

---

5. A ball is thrown from the origin of a coordinate system. The equation of its path is $y = mx - \frac{e^{2m}}{1{,}000}x^2$, where m is positive and represents the slope of the path of the ball at the origin.

   (a) For what value of m will the ball strike the horizontal axis at the greatest distance from the origin? Justify your answer.

   (b) For what value of m will the ball strike at the greatest height on a vertical wall located 100 feet from the origin?

---

6. Given two functions f and g defined by $f(x) = \tan x$ and $g(x) = \sqrt{2}\cos x$.

   (a) Find the coordinates of the point of intersection of the graphs of f and g in the interval $0 < x < \frac{\pi}{2}$.

   (b) Find the area of the region enclosed by the Y-axis and the graphs of f and g.

---

7. The rate of change in the number of bacteria in a culture is proportional to the number present. In a certain laboratory experiment, a culture had 10,000 bacteria initially, 20,000 bacteria at time t_1 minutes, and 100,000 bacteria at $(t_1 + 10)$ minutes.

   (a) In terms of t only, find the number of bacteria in the culture at any time t minutes, $t \geq 0$.

   (b) How many bacteria were there after 20 minutes?

   (c) How much time elapsed when the 20,000 bacteria were observed?

END OF EXAMINATION

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Advanced Placement Examination

MATHEMATICS: CALCULUS BC

SECTION II

Time—1 hour and 30 minutes

SHOW ALL YOUR WORK. INDICATE CLEARLY THE METHODS YOU USE BECAUSE YOU WILL BE GRADED ON THE CORRECTNESS OF YOUR METHODS AS WELL AS ON THE ACCURACY OF YOUR FINAL ANSWERS.

1. Given $f(x) = |\sin x|$, $-\pi \leq x \leq \pi$, and $g(x) = x^2$ for all real x.
   (a) On the axes provided sketch the graph of f.
   (b) Let $H(x) = g(f(x))$. Write an expression for $H(x)$.
   (c) Find the domain and range of H.
   (d) Find an equation of the line tangent to the graph of H at the point where $x = \frac{\pi}{4}$.

---

2. Let f be a function defined for all $x > -5$ and having the following properties.
   (i) $f''(x) = \frac{1}{3\sqrt{x+5}}$ for all x in the domain of f.
   (ii) The line tangent to the graph of f at (4, 2) has an angle of inclination of 45°.
   Find an expression for $f(x)$.

---

3. Let R be the region bounded by the graph of $y = \ln x$, the line $x = e$, and the X-axis.
   (a) Find the volume generated by revolving R about the X-axis.
   (b) Find the volume generated by revolving R about the Y-axis.

---

4. A ball is thrown from the origin of a coordinate system. The equation of its path is $y = mx - \frac{e^{2m}}{1{,}000}x^2$, where m is positive and represents the slope of the path of the ball at the origin.
   (a) For what value of m will the ball strike the horizontal axis at the greatest distance from the origin? Justify your answer.
   (b) For what value of m will the ball strike at the greatest height on a vertical wall located 100 feet from the origin?

5. Given the parametric equations $x = 2(\theta - \sin\theta)$ and $y = 2(1 - \cos\theta)$.
   (a) Find $\frac{dy}{dx}$ in terms of θ.
   (b) Find an equation of the line tangent to the graph at $\theta = \pi$.
   (c) Find an equation of the line tangent to the graph at $\theta = 2\pi$.
   (d) Set up but do not evaluate an integral representing the length of the curve over the interval $0 \leq \theta \leq 2\pi$. Express the integrand as a function of θ.

---

6. Given the differential equation $y'' + 4y' + 5y = 0$.
   (a) Find the general solution $y(x)$.
   (b) Find the solution that satisfies the conditions $y(0) = 0$ and $y'(0) = 1$.
   (c) Evaluate the limit of the solution determined in (b) as x increases without bound. Justify your answer.

---

7. (a) Consider a function f such that $f'(x)$ exists for all real x. Suppose that
   (i) $f'(r) = 0$ and $f'(s) = 0$,
   (ii) $f'(x) \neq 0$ for all x in the interval $r < x < s$; that is, r and s are consecutive roots of $f'(x) = 0$.
   Prove that there is at most one root of $f(x) = 0$ in $r < x < s$.
   (b) Given a function f such that $f'(x) = (4 - x^2)(4 + x^2)e^{-x^2}$. What is the maximum number of real roots of $f(x) = 0$ in $-2 < x < 2$? Justify your answer.

END OF EXAMINATION

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